NEW EQUATIONS FOR THE PROBABILISTIC PREDICTION OF SHIP ROLL MOTION IN A REALISTIC STOCHASTIC SEAWAY

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ABSTRACT

In the present work new equations are derived governing the joint, response-excitation, probability density function $f_{x(t)\dot{x}(t)y(t)}(a_1, a_2, b)$ of roll motion $x(t)$, roll velocity $\dot{x}(t)$ and excitation $y(t)$, for a ship sailing in a seaway, without any simplifying assumptions concerning the correlation and probabilistic structure of the excitation. Both external excitation, due to wind and waves, and parametric excitation, due to varying restoring coefficient, are considered. The derivation of new equations is based on Hopf's characteristic functional approach. These equations are compared (after taking the marginal with respect to the excitation) with the corresponding classical Fokker-Planck-Kolmogorov equations obtained under the assumption of delta-correlated excitation. Techniques for the numerical solution of the new equations are under development and will be presented in the near future.

Keywords: *Ship roll motion; stochastic modeling of non-linear dynamical systems; stochastic excitation; stochastic parametric excitation; non-Markovian responses; generalized Fokker-Planck-Kolmogorov equation; characteristic functional approach*

1. INTRODUCTION

Wave and wind loads on ships and structures in the sea are very successfully modelled as stochastic processes. Wave loads on ships can be considered as Gaussian or nearly Gaussian, smoothly-correlated, stochastic processes. Wind velocity and wind loads, also important for roll motion, can be considered as superposition of a steady mean and two randomly fluctuating components; one modelling the background turbulent wind flow, which is nearly stationary and nearly Gaussian with a broad-band spectrum (Simiu and Scanlan 1986, Ch. 14; Belenky and Sevastianov 2003, Sec. 8.2.1), and a second one, modelling squalls, which should be considered non-stationary and non-Gaussian (see, e.g., Belenky and Sevastianov 2003, Sec. 8.2.2, Michelacci 1983). Concerning ship motion responses, the determination of their probabilistic characteristics is straightforward as far as the assumption of linearity is (approximately) valid, and the excitation can be simplified as Gaussian. See, e.g., Price and Bishop (1974). When strong nonlinearities are present, and/or the exaction cannot be considered Gaussian, as in the case of roll motion, probabilistic characterization of the response is a difficult problem, calling for specific modelling techniques and advanced mathematical tools.

The rolling motion is perhaps the degree of freedom of ship dynamics attracted the most attention. This is perfectly justified since roll motion is easily excited in the sea, most pronounced, highly nonlinear and most dangerous; see, e.g., Belenky and Sevastianov (2003). The complicacies of the dynamics of roll motion are due partly to the nonlinearities in the restoring moment term and the damping term, and partly to the excitation mechanisms, which include external excitation by waves and wind, as well as parametric excitation. The latter is usually expressed by means of one or more time-varying (stochastic) coefficient(s), multiplying roll motion or roll velocity. The origin of parametric excitation is attributed either to heave-roll/pitch-roll nonlinear interactions (see, e.g. Oh, Nayfeh and Mook 2000, Neves and Rodriquez 2007, and references cited there), or/and to surge-roll nonlinear interaction (Spyrou 2000), or to a quasi-static variation of the GZ curve as the ship is sailing in waves (Belenky and Sevastianov 2003, Bulian 2005).

Although the investigation of nonlinear roll response under harmonic wave excitation (external or parametric) is an indispensable tool for the understanding of ship dynamics (see, e.g., Belenky and Sevastianov 2003, Bulian 2005, and the survey by Spyrou 2005), we should not forget that a ship operates in the sea and is subjected to real-world wave and wind excitation. In this paper we focus on the latter situation, aiming at the derivation of an appropriate probabilistic reformulation of the roll problem for ships sailing under the influence of irregular external or parametric excitation. The theory to be developed will cover the general case of any smoothly-correlated stochastic excitation, stationary or non-stationary, Gaussian or non-Gaussian.

One of the best ways to study the realistic roll response in the sea is to formulate (and solve) an equation governing the evolution of the joint probability density function (pdf)([1]) of roll motion and roll velocity. Early attempts in this direction, made by Haddara (1974, 1983), Haddara and Zhang (1994), Muhuri (1980), and others, were based on the assumption that the excitation (external or parametric) could be considered as a Gaussian, white-noise (delta-correlated) process. Under this assumption the roll response becomes a Markov (diffusion) process, and the arsenal of Itô calculus and Itô SDEs is available. Thus, it is possible to derive a FPK equation, describing the evolution of the joint pdf of roll motion and roll velocity.

As has been already mentioned, the assumption of Gaussian, delta-correlated excitation is rather unrealistic for a ship sailing in the sea, under the combined effect of wind and waves. In fact, the model of a delta-correlated excitation can be applied only when the correlation time of the real excitation is much smaller than the relaxation time of the system responses (see, e.g. Lin 1986, Roberts and Spanos 1986), which is not the case for the ship roll problem.

In many cases, but not always, it is possible to overcome this difficulty by using a specific technique, known as the stochastic averaging method. This method was first introduced in early sixties (Stratonovitch 1963) and made rigorous, under clearly stated assumptions, in 1966 by Khasminkii. The main feature of the method is to find a characteristic quantity of the oscillation, e.g. the amplitude of the response envelope or the total energy of the system, which varies much slower than the fluctuation of the stochastic excitation. Then, this slowly varying quantity can be considered as a Markov process, satisfying an Itô SDE. The last step of this approach consists of the calculation of the coefficients of the Itô SDE in terms of the dynamical parameters of the system and the correlation structure of the initial stochastic excitation. Among the very extensive relevant literature, we refer to the works by Ibrahim (1985), Roberts and Spanos (1986), Red-Horse and Spanos (1992), Lin an Cai (2000). See also Cai and Lin (2001), where the stochastic averaging method is compared to other methods and some of its limitations are pointed out. Variants of this methodology applied to ship rolling problem as early as 1973 (Haddara 1973, 1980, Roberts 1982). An

([1]) All abbreviations used are listed in Appendix A.

advantage of the method is that the slowly varying amplitude turns to be uncoupled with the phase processes, resulting to an easily solvable, 1+1(time) dimensional FPK equation. Furthermore, through a consideration of the related phase processes, it is possible to obtain an approximate expression for the joint (stationary) pdf of roll motion and roll velocity; see, e.g., Roberts & Dacunha (1985), Roberts (1986). The method is thoroughly re-examined in Roberts and Vasta (2000), where also an extensive survey of the previous works is presented. Further elaboration of the stochastic averaging method for the study of large ship roll responses in head irregular seas has been presented by Kreuzer and Sichermann (2007).

Another way to model satisfactorily the wave excitation due to a realistic Gaussian sea, while keeping a close connection with the standard mathematical techniques of Itô SDEs and FPK equation, is by augmenting the dynamical equations with a linear filter, excited by a delta-correlated process and providing as output a process modelling the realistic excitation. Filter description of wave field and wave excitation have been developed by Spanos (1983, 1986), and other authors, and have been applied to the model-ling and analysis of ship roll motion by Francescutto and Naito (2004) and others. The method is rather general and effective as far as the excitation is Gaus-sian. A disadvantage of this method is that the dimensionality of the augmented system (initial dynamical system plus the filter) becomes high, making very difficult the numerical solu-tion of the corresponding FPK equation. For example, the augmented system for the roll motion obtained by Francescutto and Naito (2004) has 6 degrees of freedom and, thus, the corresponding FPK equation has 6+1 (time) independent variables. Nevertheless, this formulation can be used for the systematic derivation of moment equations, providing valuable information about roll dynamics.

Apart from the above techniques for modelling the problem of the probabilistic response of a stochastically excited dynamical system, another possibility has been recently pointed out by Athanassoulis and Sapsis (2006). It is based on a generic approach introduced in 1952 by Eberhard Hopf, which treats the evolution of the underlying, infinite-dimensional, probability measure, associated with the involved processes, by means of the evolution of their joint characteristic functional (Ch.Fnl). The very demanding mathematical apparatus of this method is the price to be paid for the ability to treat any kind of stochastic processes (especially, non-Gaussian, smoothly-correlated ones). The Ch.Fnl approach has been extensively used in the statistical modelling and analysis of turbulent flows (see, e.g., Lewis & Kraichnan 1962, Beran 1968, Vishik and Fursikov 1988). The application of this approach to treat stochastically excited ODEs was discussed by Beran (1968), Ch.3. Little progress, however, has been reported in this direction, since most authors tend to avoid the use of the, somewhat obscure, concepts from infinite-dimensional calculus like Ch.Fnl. and FDEs. A notable exception is the work of Kotulski and Sobczyk (1984), who presented a closed form solution for the Ch.Fnl of a stochastically excited linear oscillator and other linear problems. In the present paper, the Ch.Fnl approach is exploited along the lines introduced by Sapsis and Athanassoulis (2008), in order to obtain new PDEs governing the evolution of the joint, response-excitation, ch.f and pdf of roll response (motion and velocity) and excitation (either external or parametric). Thus, the functional calculus and FDEs part of the analysis is used only for discovering the new PDEs which, although more complicated than the usual FPK equation, seem to be amenable to numerical treatment. Techniques for numerical solution of these equations are under development by the same authors and they will be presented in the near future.

2. THE FUNDAMENTAL DYNAMICS OF SHIP ROLL MOTION

Roll motion is one of the six, in general fully coupled, degrees of freedom of the freely floating ship. Various complicacies of the system (non-linearities, memory effects ect.) call for simplified dynamical models, in order to make feasible an in-depth analytical investigation of the problem. Our analysis will be based on the following simplified, archetypal equation of roll motion, already extensively used in the literature (see, e.g., Belenky and Sevastianov 2003):

$$\begin{aligned}(I+A)\ddot{x}(t)+b_1\dot{x}(t)+b_3\dot{x}^3(t)+\\ +\left(K_1+Y_2(t)\right)x(t)+K_3x^3(t) = Y_1(t),\end{aligned} \quad (1)$$

where $I+A$ is the total moment of inertia of the rolling ship, b_1 and b_3 are damping coefficients, K_1 and K_3 are hydrostatic coefficients, $Y_1(t)$ is the external stochastic excitation, due to the combined action of wind and waves, and $Y_2(t)$ is the parametric stochastic excitation, due to the variation of the righting arm GZ and/or to the nonlinear couplings, as discussed in the Introduction. More general model equations, including higher-order non-linear terms and additional parametric excitation terms can also be treated using the present method.

To apply our approach to eq. (1) it is expedient to rewrite it in the state-space format. Denoting $x(t)$ by $x_1(t)$ and $\dot{x}(t)$ by $x_2(t)$, eq. (1) takes the form of the following system:

$$\dot{x}_1(t)=x_2(t), \quad (2a)$$

$$\begin{aligned}\dot{x}_2(t)=b_1^{(1)}x_1(t)+b_1^{(3)}x_1^3(t)+b_2^{(1)}x_2(t)+b_2^{(3)}x_2^3(t)+\\ +b_{12}^{(1,1)}x_1(t)y_2(t;\theta)+b_{01}^{(\cdot,1)}y_1(t;\theta),\end{aligned} \quad (2b)$$

where $b_1^{(1)}=-K_1/(I+A)$, $b_1^{(3)}=-K_3/(I+A)$, $b_2^{(1)}=-b_1/(I+A)$, $b_2^{(3)}=-b_3/(I+A)$, $b_{12}^{(1,1)}=-1/(I+A)$, $b_{01}^{(\cdot,1)}=1/(I+A)$.

3. THE CHARACTERISTIC FUNCTIONAL APPROACH

The Ch.Fnl introduced in probability theory by A.N. Kolmogorov in 1935 and exploited for the first time for studying real-world problems by Hopf (1952), in connection with the description of turbulent flows. A complete mathematical description of background concepts can be found in Vakhania *et al* (1987).

3.1. A Heuristic Introduction to the Characteristic Functional

Before entering into the formal derivation of the new equations using Ch.Fnl, we shall give a heuristic motivation for that concept. If

$$P_{x(t_1)\dots x(t_N)}(d\alpha_1\dots d\alpha_N)=f_{x(t_1)\dots x(t_N)}d\alpha_1\dots d\alpha_N \quad (3)$$

is the N^{th}-order probability measure of the process $x(t)$, the corresponding ch.f is defined by the equation

$$\varphi_{x(t_1)\dots x(t_N)}(u_1,\dots,u_N)=\int_{\mathbb{R}^N}\exp\left\{i\left(u_1\alpha_1+\dots+u_N\alpha_N\right)\right\}P_{x(t_1)\dots x(t_N)}\left(d\alpha_1\dots d\alpha_N\right) \quad (4)$$

where $P_{x(t_1)\dots x(t_N)}(d\alpha_1\dots d\alpha_N)$ is the probability that the random vector $\left(x(t_1),\dots,x(t_N)\right)$ belongs to the $N-$dimensional interval $[\alpha_1,\alpha_1+d\alpha_1)\times\dots\times[\alpha_N,\alpha_N+d\alpha_N)$ at the given set of N time instants $t_1,\dots,t_N$. Now, following the concept of passing from the discrete to continuous, introduced by V. Volterra in 1887 (see, also, Volterra 1930/1959/2005), we consider that $N\to\infty$ and $t_n-t_{n-1}\to 0$, so that t_n's tend to become densely distributed in an interval, say $[t_0,T]$, and the sum $u_1\alpha_1+\dots+u_N\alpha_N$ is replaced by the integral $\int_{t_0}^{T}u(t)\alpha(t)\,dt$. The limiting form of the ch.f, eq. (4), is the Ch.Fnl of the process $x(t),\ t\in[t_0,T]$, denoted by $\mathscr{F}_x(u)$. Intensive mathematical research of the last four decades has established that Ch.Fnl is well defined as a new kind of integral over infinite-dimensional spaces, retaining most of the essential properties of the usual integral (Vakhania *et al* 1987, Egorov *et al* 1993). Based on the above discussion we can write the following heuristic representation for $\mathscr{F}_x(u)$:

$$\mathscr{F}_x(u)=\int_{\mathbb{R}^{[t_0,T]}}\exp\left\{i\int_{t_0}^{T}u(t)\alpha(t)\,dt\right\}\mathscr{P}_x(d\alpha). \quad (5)$$

In the above equation, $u(t)$, $\alpha(t)$ and $d\alpha=d\alpha(t)$ are functions defined on the whole interval $[t_0,T]$, $\mathbb{R}^{[t_0,T]}$ denotes the set of functions

$\alpha:[t_0,T]\to\mathbb{R}$, and $\mathscr{P}_x(d\alpha)$ is the probability that a sample function $x(t),\ t\in[t_0,T]$, lies in the "strip" between $\alpha(t)$ and $\alpha(t)+d\alpha(t)$, for all $t\in[t_0,T]$. Thus, the exterior integration in eq. (5) is considered over all functions $\alpha(t)$ from the sample space of the process. We shall denote the sample (function) space by $\mathscr{X}$ and will assume that $\mathscr{X}$ is either $C([t_0,T])$, the space of all continuous functions defined on $[t_0,T]$, or $C^1([t_0,T])$, the space of differentiable functions defined on $[t_0,T]$. From the mathematical point of view an even more general choice is technically preferable: $u(\cdot)$ will be considered as a generalized function (continuous functional) over the sample-function space $\mathscr{X}$. The space of all continuous functionals over $\mathscr{X}$ (the topological dual of $\mathscr{X}$) will be denoted by $\mathscr{U}$. On the basis of the above discussion, the integral appearing in the argument of the exponential in eq. (5) is replaced by the symbol (duality pairing) $<\boldsymbol{u},\boldsymbol{\alpha}>$, which denotes the evaluation of the continuous functional $\boldsymbol{u}$ against the function $\boldsymbol{\alpha}$. After these explanations we can introduce the following, formal, definition of the Ch.Fnl of a stochastic process $x(t),\ t\in[t_0,T]$:

$$\mathscr{F}_x(\boldsymbol{u})=\int_{\mathscr{X}}\exp\{i<\boldsymbol{u},\boldsymbol{\alpha}>\}\mathscr{P}_x(d\boldsymbol{\alpha}). \qquad (6)$$

In treating the stochastic response of dynamical system (2), we have to consider the state response process $(x_1(t),x_2(t))$, as well as of the joint, response-excitation, process $(x_1(t),x_2(t), y_1(t),y_2(t))$. Thus, we have to consider the joint Ch.Fnl $\mathscr{F}_{x_1x_2y_1y_2}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2)$, abbreviated to $\mathscr{F}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2)$, and defined by

$$\mathscr{F}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v}_1,\boldsymbol{v}_2)=$$

$$\iint_{\mathscr{X}_1\times\mathscr{X}_2}\iint_{\mathscr{Y}_1\times\mathscr{Y}_2}\exp\{i(<\boldsymbol{u}_1,\boldsymbol{\alpha}_1>+<\boldsymbol{u}_2,\boldsymbol{\alpha}_2>+$$

$$+<\boldsymbol{v}_1,\boldsymbol{\beta}_1>+<\boldsymbol{v}_2,\boldsymbol{\beta}_2>)\}\mathscr{P}(d\boldsymbol{\alpha}_1d\boldsymbol{\alpha}_2d\boldsymbol{\beta}_1d\boldsymbol{\beta}_2). \ (7)$$

In the above equation $\mathscr{P}(d\boldsymbol{\alpha}_1d\boldsymbol{\alpha}_2d\boldsymbol{\beta}_1d\boldsymbol{\beta}_2)=\mathscr{P}_{x_1x_2y_1y_2}(d\boldsymbol{\alpha}_1d\boldsymbol{\alpha}_2d\boldsymbol{\beta}_1d\boldsymbol{\beta}_2)$ is the joint probability measure for the vector process $(x_1(t),x_2(t), y_1(t),y_2(t)),\ t\in[t_0,T]$, $\mathscr{X}_1$ and $\mathscr{X}_2$ are the sample-function spaces for the processes $x_1(t)$ and $x_2(t)$, respectively, both assumed to be $C^1([t_0,T])$, whereas $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are the sample-function spaces for the external and parametric excitation processes $y_1(t),y_2(t)$, respectively, both taken as $C([t_0,T])$. To avoid lengthy and cumbersome mathematical expressions we shall restrict ourselves to the cases:

- $y_1(t)\neq 0,\ y_2(t)=0$ (external excitation), and
- $y_1(t)=0,\ y_2(t)\neq 0$ (parametric excitation).

Then, denoting the non-zero excitation as $y(t)$, we shall work with the the joint, response-excitation Ch.Fnl.

$$\mathscr{F}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v})=\iiint\exp\{\cdots\}\mathscr{P}(d\boldsymbol{\alpha}_1d\boldsymbol{\alpha}_2d\boldsymbol{\beta}) \ (8)$$

where

$$\exp\{\cdots\}=\exp\{i(<\boldsymbol{u}_1,\boldsymbol{\alpha}_1>+<\boldsymbol{u}_2,\boldsymbol{\alpha}_2>+<\boldsymbol{v},\boldsymbol{\beta}>)\}.$$

The general case $y_1(t)\neq 0,\ y_2(t)\neq 0$ can be treated in a similar way.

4. HOPF'S FUNCTIONAL DIFFERENTIAL EQUATION

Following the methodology of Hopf (1952) (see also Beran 1968), we shall first derive FDEs for the joint Ch.Fnl. $\mathscr{F}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{v})$. For this purpose use will be made of appropriate functional derivatives. (See Appendix B for relevant definitions and formulae.) Applying formula (45), Appendix B, with $h_{u_1}=\delta_t(\cdot)$, the Dirac delta functional supported at t, we obtain

$$\delta_{u_1}\mathscr{F}=\iint i\,\alpha_1(t)\cdot\exp\{\cdots\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}), \qquad (9)$$

$$\delta_{u_2}\mathscr{F}=\iint i\,\alpha_2(t)\cdot\exp\{\cdots\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}), \qquad (10)$$

$$\delta_{v}\mathscr{F}=\iint i\,\beta(t)\cdot\exp\{\cdots\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}), \qquad (11)$$

where $d\boldsymbol{\alpha}=d\alpha_1d\alpha_2$. Differentiating now with respect to time, and assuming that time derivative $d(\cdot)/dt$ can pass under the (functional) integral sign, we obtain:

$$\frac{1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F}=\iint\dot{\alpha}_1(t)\cdot\exp\{\cdots\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}), \qquad (12)$$

$$\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}=\iint\dot{\alpha}_2(t)\cdot\exp\{\cdots\}\mathscr{P}_{xy}(d\boldsymbol{\alpha}\,d\boldsymbol{\beta}). \qquad (13)$$

In a similar manner, applying formulae (47), Appendix B, we get:

$$\frac{1}{i^3}\,\delta^{(3)}_{u_1}\mathscr{F} = \iint \alpha_1^3(t)\cdot\exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta), \quad (14)$$

$$\frac{1}{i^3}\,\delta^{(3)}_{u_2}\mathscr{F} = \iint \alpha_2^3(t)\cdot\exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta). \quad (15)$$

Combining now the differential system (2) with $y_2(t)=0$ and $y_1(t)\triangleq y(t)$, with the functional derivatives (9)-(15), we obtain:

$$\frac{1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F} - \frac{1}{i}\delta_{u_2}\mathscr{F} = \\ = \iint\left(\dot{\alpha}_1(t)-\alpha_2(t)\right)\exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta) = 0 \quad (16)$$

and

$$\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F} - \frac{1}{i}b_1^{(1)}\delta_{u_1}\mathscr{F} - \frac{1}{i^3}b_1^{(3)}\delta^{(3)}_{u_1}\mathscr{F} \\ -\frac{1}{i}b_2^{(1)}\delta_{u_2}\mathscr{F} - \frac{1}{i^3}b_2^{(3)}\delta^{(3)}_{u_2}\mathscr{F} - \frac{1}{i}b_{01}^{(\cdot,1)}\delta_{\upsilon}\mathscr{F} = \\ = \iint\left(\dot{\alpha}_2(t) - b_1^{(1)}\alpha_1(t) - b_1^{(3)}\alpha_1^3(t) - b_2^{(1)}\alpha_2(t)\right. \\ \left. - b_2^{(3)}\alpha_2^3(t) - b_{01}^{(\cdot,1)}\beta(t)\right)\times\exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta) = 0\ .(17)$$

The system (16), (17) is supplemented by the marginal-compatibility condition:

$$\mathscr{F}(0,0,\upsilon) \equiv \mathscr{F}_{x_1x_2y}(0,0,\upsilon) = \mathscr{F}_y(\upsilon) \quad (18)$$

Also, appropriate initial conditions are needed, to ensure that the initial state $x_{n0}(\theta)$, $n=1,2$, is probabilistically given and independent of the excitation $y(t)$. To this end we set $\boldsymbol{u}_n = \tilde{u}_n\delta_{t_0}(\cdot)$, $\tilde{u}_n \in \mathbb{R}$, $n=1,2$, where $\delta_{t_0}(\cdot)\in\mathscr{U}$ is the Dirac delta functional supported at t_0 and, invoking the projection theorem (see Sapsis and Athanassoulis 2008), we obtain:

$$\mathscr{F}\left(\tilde{u}_1\delta_{t_0}(\cdot), \tilde{u}_2\delta_{t_0}(\cdot), 0\right) = \varphi_{x_1(t_0)x_2(t_0)}\left(\tilde{u}_1,\tilde{u}_2\right), \quad (19)$$

where $\varphi_{x_1(t_0)x_2(t_0)}(\tilde{u}_1,\tilde{u}_2)$ is the (known) joint ch.f. of the initial state $(x_{10}(\theta), x_{20}(\theta))$.

5. EVOLUTION EQUATION FOR THE JOINT, RESPONSE-EXCITATION, pdf UNDER GENERAL EXTERNAL EXCITATION

In order to take an equation for the joint, response-excitation,ch.f we proceed as follows: Consider $(\tilde{u}_1, \tilde{u}_2)\in\mathbb{R}^2$, and take the linear combination of the left-hand sides of equations (16), (17), multiplied by $i\tilde{u}_1$, $i\tilde{u}_2$ respectively:

$$i\tilde{u}_1\left(\frac{1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F} - \frac{1}{i}\delta_{u_2}\mathscr{F}\right) + i\tilde{u}_2\left(\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}\right. \\ -\frac{1}{i}b_1^{(1)}\delta_{u_1}\mathscr{F} - \frac{1}{i^3}b_1^{(3)}\delta^{(3)}_{u_1}\mathscr{F} - \frac{1}{i}b_2^{(1)}\delta_{u_2}\mathscr{F} \\ \left. -\frac{1}{i^3}b_2^{(3)}\delta^{(3)}_{u_2}\mathscr{F} - \frac{1}{i}b_{01}^{(\cdot,1)}\delta_{\upsilon}\mathscr{F}\right) = 0. \quad (20)$$

In the arguments of $\mathscr{F}=\mathscr{F}(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{\upsilon})$ we make the substitution $\boldsymbol{u}_1 = \tilde{u}_1\delta_t(\cdot)$, $\boldsymbol{u}_2 = \tilde{u}_2\delta_t(\cdot)$, $\boldsymbol{\upsilon} = \tilde{\upsilon}\delta_s(\cdot)$, $\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}\in\mathbb{R}$, where t,s are two arbitrary time instants (later on we shall take $t\to s$). Then, for the linear combination of the time derivatives we have:

$$\left(i\tilde{u}_1\frac{1}{i}\frac{d}{dt}\delta_{u_1}\mathscr{F} + i\tilde{u}_2\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}\right)\Bigg|_{\substack{\boldsymbol{u}_1=\tilde{u}_1\delta_t(\cdot)\\ \boldsymbol{u}_2=\tilde{u}_2\delta_t(\cdot)\\ \boldsymbol{\upsilon}=\tilde{\upsilon}\delta_s(\cdot)}} = \\ = \iint\left(i\tilde{u}_1\dot{\alpha}_1(t) + i\tilde{u}_2\dot{\alpha}_2(t)\right)\exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta) = \\ = \frac{\partial}{\partial t}\iint \exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta) =$$

[invoking once again the projection theorem]

$$= \frac{\partial}{\partial t}\int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}}\exp\{\cdots\}\,P_{x_1(t)x_2(t)y(s)}\left(d\alpha_1 d\alpha_2 d\beta\right) = \\ = \frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}\right), \quad (21)$$

where:

$$\exp\{\cdots\} = \exp\left\{i\left(\tilde{u}_1\alpha_1 + \tilde{u}_2\alpha_2 + \tilde{\upsilon}\beta\right)\right\}. \quad (22)$$

The remaining terms appearing in equation (20) can also be transformed in a similar way and reduced to their finite-dimensional counterparts:

$$\frac{1}{i}\delta_{u_2}\mathscr{F}\Bigg|_{\substack{\boldsymbol{u}_1=\tilde{u}_1\delta_t(\cdot)\\ \boldsymbol{u}_2=\tilde{u}_2\delta_t(\cdot)\\ \boldsymbol{\upsilon}=\tilde{\upsilon}\delta_s(\cdot)}} = \iint \alpha_2(t)\exp\{\cdots\}\,\mathscr{P}_{xy}(d\alpha\, d\beta) \\ = \frac{1}{i}\frac{\partial}{\partial\tilde{u}_2}\int_{\mathbb{R}^{(2)}}\int_{\mathbb{R}}\exp\{\cdots\}\,P_{x_1(t)x_2(t)y(s)}\left(d\alpha_1 d\alpha_2 d\beta\right) = \\ = \frac{1}{i}\frac{\partial}{\partial\tilde{u}_2}\varphi_{x_1(t)x_2(t)y(s)}, \quad (23)$$

where

$$\varphi_{x_1(t)x_2(t)y(s)} = \varphi_{x_1(t)x_2(t)y(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{\upsilon}\right),$$

$$\frac{1}{i^3}\delta^{(3)}_{u_1}\mathscr{F}\Bigg|_{\substack{u_1=\tilde{u}_1\delta_t(\cdot)\\ u_2=\tilde{u}_2\delta_t(\cdot)\\ v=\tilde{v}\delta_s(\cdot)}}=\frac{1}{i^3}\frac{\partial^3}{\partial\tilde{u}_1^3}\varphi_{x_1(t)x_2(t)y(s)}, \quad (24)$$

$$\frac{1}{i}\delta_{u_1}\mathscr{F}\Bigg|_{\substack{u_1=\tilde{u}_1\delta_t(\cdot)\\ u_2=\tilde{u}_2\delta_t(\cdot)\\ v=\tilde{v}\delta_s(\cdot)}}=\frac{1}{i}\frac{\partial}{\partial\tilde{u}_1}\varphi_{x_1(t)x_2(t)y(s)}, \quad (25)$$

$$\frac{1}{i}\delta_{v}\mathscr{F}\Bigg|_{\substack{u_1=\tilde{u}_1\delta_t(\cdot)\\ u_2=\tilde{u}_2\delta_t(\cdot)\\ v=\tilde{v}\delta_s(\cdot)}}=\frac{1}{i}\frac{\partial}{\partial\tilde{v}}\varphi_{x_1(t)x_2(t)y(s)}, \quad (26)$$

$$\frac{1}{i^3}\delta^{(3)}_{u_2}\mathscr{F}\Bigg|_{\substack{u_1=\tilde{u}_1\delta_t(\cdot)\\ u_2=\tilde{u}_2\delta_t(\cdot)\\ v=\tilde{v}\delta_s(\cdot)}}=\frac{1}{i^3}\frac{\partial^3}{\partial\tilde{u}_2^3}\varphi_{x_1(t)x_2(t)y(s)}. \quad (27)$$

Equations (20)-(27) are valid for any $t,s\in[t_0,T]$. Taking the limit $s\to t$ and substituting the limiting forms of eq. (21)-(27) to eq. (20), we get the following equation for the joint, response-excitation, ch.f.:

$$\frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y(s)}\Bigg|_{s\to t}-\tilde{u}_1\frac{\partial}{\partial\tilde{u}_2}\varphi_{x_1(t)x_2(t)y(t)}$$
$$-\tilde{u}_2\Bigg(b_1^{(1)}\frac{\partial}{\partial\tilde{u}_1}\varphi_{x_1(t)x_2(t)y(t)}+\frac{b_1^{(3)}}{i^2}\frac{\partial^3}{\partial\tilde{u}_1^3}\varphi_{x_1(t)x_2(t)y(t)}$$
$$+b_2^{(1)}\frac{\partial}{\partial\tilde{u}_2}\varphi_{x_1(t)x_2(t)y(t)}+\frac{b_2^{(3)}}{i^2}\frac{\partial^3}{\partial\tilde{u}_2^3}\varphi_{x_1(t)x_2(t)y(t)}$$
$$+b_{01}^{(\cdot,1)}\frac{\partial}{\partial\tilde{v}_1}\varphi_{x_1(t)x_2(t)y(t)}\Bigg)=0. \quad (28)$$

Since the marginal ch.f. related to the excitation is known, the following compatibility condition should hold true:

$$\varphi_{x_1(t)x_2(t)y(s)}\left(0,0,\tilde{v}_1\right)=\varphi_{y(s)}\left(\tilde{v}_1\right)=$$
$$=\textit{known ch.f.},\quad \forall s\geq t_0. \quad (29)$$

Furthermore, making the plausible assumption that the initial state is independent from the excitation, the initial condition takes the form

$$\varphi_{x_1(t_0)x_2(t_0)}\left(\tilde{u}_1,\tilde{u}_2\right)=\textit{known ch.f.}. \quad (30)$$

Since the ch.f. $\varphi_{x_1(t)x_2(t)y(s)}\left(\tilde{u}_1,\tilde{u}_2,\tilde{v}\right)$ is the Fourier transform of pdf $f_{x_1(t)x_2(t)y(s)}\left(\alpha_1,\alpha_2,\beta\right)$, eq. (28) can be restated in terms of the pdf as follows:

$$\frac{\partial}{\partial t}f_{x_1(t)x_2(t)y(s)}\Bigg|_{s\to t}+\ b_2^{(1)}f_{x_1(t)x_2(t)y(t)}\ +$$
$$+3b_2^{(3)}\alpha_2^2 f_{x_1(t)x_2(t)y(t)}+\alpha_2\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_1}$$
$$+\ b_1^{(1)}\alpha_1\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}+b_1^{(3)}\alpha_1^3\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}$$
$$+\ b_2^{(1)}\alpha_2\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}+b_2^{(3)}\alpha_2^{\,3}\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}$$
$$+b_{01}^{(\cdot,1)}\beta\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}=0. \quad (31a)$$

The above equation will be abbreviated as

$$\frac{\partial}{\partial t}f_{x_1(t)x_2(t)y(s)}\Bigg|_{s\to t}+\mathscr{L}_{\alpha_1\alpha_2}f_{x_1(t)x_2(t)y(t)}\ +$$
$$+\ b_{01}^{(\cdot,1)}\beta\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}=0, \quad (31b)$$

where $\mathscr{L}_{\alpha_1\alpha_2}$ is a differential operator (including differentiation only with respect to α_1,α_2), the exact form of which is easily concluded by comparing the equations (31a) and (31b). Marginal-compatibility and initial conditions take now the form

$$f_{y(s)}(\beta)=\iint_{\mathbb{R}^2}f_{x_1(t)x_2(t)y(s)}\left(\alpha_1,\alpha_2,\beta\right)d\alpha_1 d\alpha_2=$$
$$=\textit{known density function},\quad \forall s\geq t_0, \quad (32)$$

$$f_{x_1(t_0)x_2(t_0)}\left(\alpha_1,\alpha_2\right)=\textit{known density function} \quad (33)$$

6. EVOLUTION EQUATION FOR THE JOINT, RESPONSE-EXCITATION, pdf UNDER STOCHASTIC PARAMETRIC EXCITATION

Consider now the case $y_1(t)=0,\ y_2(t)\neq 0$ (parametric excitation). Let now $\mathscr{F}\left(u_1,u_2,v\right)$ be the joint, response-parametric excitation, Ch.Fnl. In this case use will be made of the formula

$$\frac{1}{i^2}\delta_{u_1}\delta_{v}\mathscr{F}=\iint i\,\alpha_1(t)\beta(t)\cdot\exp\{\cdots\}\mathscr{P}_{xy}\left(d\alpha\, d\beta\right). \quad (34)$$

Combining formulae (9), (10), (12)-(15) and (34) for functional derivatives, and taking into account the differential system (2) (with $y_1(t)=0$), we obtain again equation (16), as well as the following FDE:

$$\frac{1}{i}\frac{d}{dt}\delta_{u_2}\mathscr{F}-\frac{1}{i}b_1^{(1)}\delta_{u_1}\mathscr{F}-\frac{1}{i^3}b_1^{(3)}\delta_{u_1}^{(3)}\mathscr{F}$$
$$-\frac{1}{i}b_2^{(1)}\delta_{u_2}\mathscr{F}-\frac{1}{i^3}b_2^{(3)}\delta_{u_2}^{(3)}\mathscr{F}-\frac{1}{i^2}b_{12}^{(1,1)}\delta_{u_1}\delta_{\upsilon}\mathscr{F}=$$
$$=\iint\left(\dot{\alpha}_2(t)-b_1^{(1)}\alpha_1(t)-b_1^{(3)}\alpha_1^3(t)-b_2^{(1)}\alpha_2(t)-b_2^{(3)}\alpha_2^3(t)\right.$$
$$\left.-b_{12}^{(1,1)}\alpha_1(t)\beta(t)\right)\times\exp\{\cdots\}\mathscr{P}_{xy}\left(d\boldsymbol{\alpha}d\boldsymbol{\beta}\right)=0. \quad (35)$$

The new system of FDEs (16), (35) is also be supplemented by the marginal-compatibility condition (18) and the initial condition (19).

Working similarly as in Sec. 5, and taking into account equation

$$\frac{1}{i^2}\delta_{u_1}\delta_{\upsilon}\mathscr{F}\Bigg|_{\substack{u_1=\tilde{u}_1\delta_t(\cdot)\\ u_2=\tilde{u}_2\delta_t(\cdot)\\ \upsilon=\tilde{\upsilon}\delta_s(\cdot)}}=\frac{1}{i^2}\frac{\partial}{\partial\tilde{u}_1}\frac{\partial}{\partial\tilde{\upsilon}}\varphi_{x_1(t)x_2(t)y(s)}, \quad (36)$$

we obtain the following equation for the joint, response-parametric excitation, ch.f:

$$\frac{\partial}{\partial t}\varphi_{x_1(t)x_2(t)y(s)}\Bigg|_{s\to t}-\tilde{u}_1\frac{\partial}{\partial\tilde{u}_2}\varphi_{x_1(t)x_2(t)y(t)}$$
$$-\tilde{u}_2\left(b_1^{(1)}\frac{\partial}{\partial\tilde{u}_1}\varphi_{x_1(t)x_2(t)y(t)}+\frac{b_1^{(3)}}{i^2}\frac{\partial^3}{\partial\tilde{u}_1^3}\varphi_{x_1(t)x_2(t)y(t)}+\right.$$
$$+b_2^{(1)}\frac{\partial}{\partial\tilde{u}_2}\varphi_{x_1(t)x_2(t)y(t)}+\frac{b_2^{(3)}}{i^2}\frac{\partial^3}{\partial\tilde{u}_2^3}\varphi_{x_1(t)x_2(t)y(t)}+$$
$$\left.+\frac{b_{12}^{(1,1)}}{i}\frac{\partial}{\partial\tilde{u}_1}\frac{\partial}{\partial\tilde{\upsilon}}\varphi_{x_1(t)x_2(t)y_2(s)}\right)=0\cdot \quad (37)$$

Marginal-compatibility and initial conditions (29),(30) should also hold true. Applying the inverse Fourier transform to eq. (37), we obtain the following equation for the joint, response-parametric excitation, pdf:

$$\frac{\partial}{\partial t}f_{x_1(t)x_2(t)y(s)}\Bigg|_{s\to t}+\mathscr{L}_{\alpha_1\alpha_2}f_{x_1(t)x_2(t)y(t)}+$$
$$+b_{12}^{(1,1)}\alpha_2\beta\frac{\partial f_{x_1(t)x_2(t)y(t)}}{\partial\alpha_2}=0, \quad (38)$$

where $\mathscr{L}_{\alpha_1\alpha_2}$ is the same differential operator as the one appearing in (31b). Marginal-compatibility and initial conditions (32), (33) also hold true.

7. CONCLUSIONS. COMPARISON WITH EXISTING METHODS BASED ON THE FPK EQUATION

In this paper the characteristic functional approach has been invoked in order to derive new equations governing the joint, response-excitation, ch.f and pdf of the ship's roll motion, velocity and excitation. Two cases have been considered: First, external stochastic excitation, due to the combined action of wind and waves (eqs. (28) and (31)). Second, parametric stochastic excitation, due to the variation of the righting arm GZ and/or to the nonlinear heave-roll, pitch-roll coupling (eqs. (37) and (38)). Appropriate marginal-compatibility and initial conditions are also given, in order to make these equations uniquely solvable. Techniques for the numerical solution of these equations are under development and will be presented in the near future.

In our approach both external and parametric stochastic excitation are considered to be general, smoothly-correlated, continuous-path, stochastic processes. No specific simplifying (artificial) assumptions, concerning either the correlation structure or the distributions of the stochastic data, are needed. Because of the general nature of the excitation, the roll response is generally non-Markovian.

It is expedient to compare our new equations with the ones obtained by means of the classical Itô approach (see, e.g., Haddara 1974, 1983, Haddara and Zhang 1994, Muhuri 1980), under the assumption that the excitation is a Gaussian, delta-correlated process. In the later case, it is straightforward to derive the corresponding FPK equation that governs the joint pdf $f_{x_1(t)x_2(t)}=f_{x_1(t)x_2(t)}(\alpha_1,\alpha_2)$ of the ship's roll motion and velocity. In particular, when only external stochastic excitation is considered, FPK equation takes the form:

$$\frac{\partial}{\partial t}f_{x_1(t)x_2(t)}+\mathscr{L}_{\alpha_1\alpha_2}f_{x_1(t)x_2(t)}$$
$$-\left(b_{01}^{(\cdot,1)}\right)^2D_{22}\frac{\partial^2 f_{x_1(t)x_2(t)}}{\partial\alpha_2^{\ 2}}=0, \quad (39)$$

where $2D_{22}$ is the intensity of the roll exciting moment process ($< y(t)y(t+\tau) > = 2D_{22}\delta(\tau)$).

In order to compare our new eq. (31) with eq. (39), we integrate eq. (31) with respect to β, obtaining thus the equation

$$\frac{\partial}{\partial t} f_{x_1(t)x_2(t)} + \mathscr{L}_{\alpha_1\alpha_2} f_{x_1(t)x_2(t)} + b_{01}^{(\cdot,1)} \frac{\partial}{\partial \alpha_2} \int \beta f_{x_1(t)x_2(t)y(t)}(\alpha_1,\alpha_2,\beta)\, d\beta = 0 \quad (40)$$

for the evolution of the β– marginal, (α_1,α_2)– joint pdf of roll response (joint motion-velocity pdf)

$$f_{x_1(t)x_2(t)} \equiv f_{x_1(t)x_2(t)}(\alpha_1,\alpha_2) = = \int_{\mathbb{R}} f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)\, d\beta . \quad (41)$$

Eq. (39) [classical FPK] differs from eq. (40) [generalized FPK] only in the last term which is the one connected with the stochastic excitation. The term appearing in eq. (39) is of local character (second α_2– derivative) and contains the (known) intensity of the stochastic excitation. The corresponding term in eq. (40) is non-local (integrates over the whole range of possible excitation values) and cannot be expressed only in terms of $f_{x_1(t)x_2(t)}(\alpha_1,\alpha_2)$, i.e. eq. (40) is not a closed equation. Additional information about the dependence between excitation and response should be used as a closure condition. A general method providing tractable closure conditions, through a chain of FPK-like equations, will be presented elsewhere. On the other hand, the difficulties associated with the closure disappear if we choose to work with eq. (31), solve it and find first the joint pdf $f_{x_1(t)x_2(t)y(s)}(\alpha_1,\alpha_2,\beta)$, and then calculate the marginal $f_{x_1(t)x_2(t)}(\alpha_1,\alpha_2)$.

It is also interesting to note that the classical FPK eq. can be considered as a special case of the new generalised FPK eq., also in terms of their derivation. In fact, the classical FPK can be (re)derivated by means of the characterristic functional approach applied to the finite difference version of the initial-value problem (1) under a delta-correlated, Gaussian excitation. This result has been proved for the scalar case in Sapsis and Athanassoulis (2008). The corresponding study for a system of stochastic differential equations will be presented in the near future.

Similar conclusions can beô drawn for the case of parametric stochastic excitation. The classical Itô eq. in this case takes the form

$$\frac{\partial}{\partial t} f_{x_1(t)x_2(t)} + \mathscr{L}_{\alpha_1\alpha_2} f_{x_1(t)x_2(t)} \alpha_1^{\,2}\left(b_{12}^{(1,1)}\right)^2 D_{22} \frac{\partial^2 f_{x_1(t)x_2(t)}(\alpha_1,\alpha_2)}{\partial \alpha_2^{\,2}} = 0 \quad (42)$$

whereas integrating eq. (39), we have:

$$\frac{\partial}{\partial t} f_{x_1(t)x_2(t)} + \mathscr{L}_{\alpha_1\alpha_2} f_{x_1(t)x_2(t)} + b_{12}^{(1,1)} \alpha_2 \frac{\partial}{\partial \alpha_2} \int \beta f_{x_1(t)x_2(t)y_2(t)}\, d\beta = 0 \quad (43)$$

Again the two equations differ only in their last terms, and very similar comments as in the previous case can be made.

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9. APPENDIX A: ABBREVIATIONS

ch.f	characteristic function
Ch.Fnl	characteristic functional
FDE(s)	functional differential equation(s)

FPK	Fokker-Planck-Kolmogorov (equation)
ODE(s)	ordinary differential equation(s)
PDE(s)	partial differential equation(s)
pdf	probability density function
SDE(s)	stochastic differential equation(s)

10. APPENDIX B: GATEAUX FUNCTIONAL DERIVATIVES

The first-order Gateaux functional derivative of functional $\mathscr{F}(u_1,u_2,\upsilon)$ with respect to the first variable (u_1), along the direction h_{u_1} is defined as follows (Volterra 1930/1959/2005, Beran 1968):

$$\delta_{u_1}\mathscr{F}\left(\left[u_1;h_{u_1}\right],u_2,\upsilon\right) \equiv \delta_{u_1}\mathscr{F} = \lim_{\varepsilon\to 0}\frac{d}{d\varepsilon}\mathscr{F}\left(u_1+\varepsilon h_{u_1},u_2,\upsilon\right). \tag{44}$$

To save space, the abbreviation $\delta_{u_1}\mathscr{F}$ has been used instead of $\delta_{u_1}\mathscr{F}\left(\left[u_1;h_{u_1}\right],u_2,\upsilon\right)$. Derivatives $\delta_{u_2}\mathscr{F}$ and $\delta_{\upsilon}\mathscr{F}$ are defined similarly.

Applying the definition (44) to the Ch.Fnl eq. (8), we easily obtain:

$$\delta_{u_1}\mathscr{F}\left(\left[u_1;h_{u_1}\right],u_2,\upsilon\right) = = \iint i < h_{u_1},\alpha_1 > \exp\{\cdots\}\mathscr{P}_{xy}(d\alpha\, d\beta) \tag{45}$$

where

$$\exp\{\cdots\} = \exp\left\{i\left(<u_1,\alpha_1> + <u_2,\alpha_2> + <\upsilon,\beta>\right)\right\}.$$

Applying the eq. (45) to $h_{u_1} = \delta_t(\cdot)$, the Dirac delta functional supported at t, we get eq. (9). Eqs (10) and (11) are obtained similarly.

Assume now that q_1 is a non-negative integer, and apply q_1 times consecutively Gateaux differentiation to functional $\mathscr{F}(u_1,u_2,\upsilon)$ wrt to the first functional variable u_1, in the directions $h_{u_1}^{(1)},h_{u_1}^{(2)},\ldots,h_{u_1}^{(q_1)}$. This is the q_1 – fold Gateaux derivative of $\mathscr{F}(u_1,u_2,\upsilon)$. Especially for the Ch.Fnl (8), the q_1 – fold Gateaux derivative wrt u_1 in the directions $h_{u_1}^{(1)},h_{u_1}^{(2)},\ldots,h_{u_1}^{(q_1)}$ is given by the formula

$$\delta_{u_1}^{(q_1)}\mathscr{F} = \delta_{u_1}^{(q_1)}\mathscr{F}\left(\left[u_1;h_{u_1}^{(1)},\ldots,h_{u_1}^{(q_1)}\right],u_2,\upsilon\right) = =\iint i^{q_1}<h_{u_1}^{(1)},x_1>\ldots<h_{u_1}^{(q_1)},x_1>\exp\{\cdots\}\mathscr{P}_{xy}(d\alpha\, d\beta). \tag{46}$$

Applying the above general formula to the directions $h_{u_1}^{(1)} = h_{u_1}^{(2)} = h_{u_1}^{(3)} = \delta_t(\cdot)$, we find

$$\delta_{u_1}^{(3)}\mathscr{F} = \delta_{u_1}^{(3)}\mathscr{F}\left(\left[u_1;\delta_t(\cdot),\delta_t(\cdot),\delta_t(\cdot)\right],u_2,\upsilon\right) = = \int\int i^3 x_1^3(t)\exp\{\cdots\}\mathscr{P}_{xy}(d\alpha\, d\beta). \tag{47}$$

In a similar manner, we can define the (q_1,q_2,q_3) – fold mixed Gateaux derivative, q_1 – fold wrt to u_1, q_2 – fold wrt to u_2 and q_3 – fold wrt to υ, assuming that all these derivatives do exist.