On the Distribution of Parametric Roll

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ABSTRACT

It is known that the distribution of a ship's roll angle during parametric resonance is far from Gaussian. The most pronounced feature is an excess of kurtosis, making the distribution very "peaked" at the mean value. Large kurtosis also means "fat" tails, which may be a direct result of the periods of large roll present in the time-series.

This paper considers several alternatives for modeling the distribution, including a Gram-Charlier series, the Pearson type IV distribution, and an approximation based on a moving average. Special attention is paid to the modeling of the tails of the distribution, as they may contain information on "how far the ship can go". The peak-over-threshold method is considered for processing the very scarce data available for large roll angles, with the idea of using it to model the behavior of the tail of the distribution of parametric roll.

KEYWORDS

Parametric roll, probability density distribution

INTRODUCTION

Significant interest to parametric roll was renewed in the late 1990s, when large container carriers were found to be susceptible to parametric roll resonance (France et al. 2003). However, the physical phenomenon of parametric roll resonance had been known for almost 70 years (Kempf 1938, Graff and Heckscher 1941, Kerwin 1955, Paulling and Rosenberg 1959).

Parametric roll in irregular waves presents a serious challenge for both theoretical and numerical analysis. Nonlinearity of dynamical system manifests itself as a dependence of the frequency range on amplitude (Spyrou 2000, Bulian et al. 2004, Neves and Rodríguez 2007), a probability distribution that is far from normal (Belenky et al. 2003, Hashimoto et al. 2006, Umeda et al. 2008), and indications that the assumption of ergodicity is not applicable for practical purposes (Belenky 2004, Bulian et al. 2006). Nevertheless, estimates of statistical characteristics are necessary in order to develop ship-specific operational guidance (ABS 2004). These estimates are performed using a Monte-Carlo time-domain numerical simulation (Shin et al. 2004, Belenky et al. 2006).

Belenky and Weems (2011) examined the uncertainty of record and ensemble estimates of mean values and variance. It was shown that, because of "practical" non-ergodicity, the difference between the values estimated on two records of reasonable length could be quite significant, so several records are needed. Some convergence was observed for an ensemble of 5 records of 25 minutes each, while stable results required 12-20 such records.

The next challenge is to find a distribution that fits parametric roll. The analysis is performed on the dataset from Belenky and Weems (2011), which consists of 50 records of 25 minutes duration each. The data was obtained by numerical simulation of the motions of a C11-class container carrier in long-crested head seas with a forward speed of 10 kn. Numerical simulations were carried out with the advanced hybrid code LAMP (Shin et al. 2003). Three degrees of freedom were modeled: heave, roll, and pitch. The hydrodynamic formulation included a body-nonlinear calculation of the

Froude-Krylov and hydrostatic forces and a solution of a body linear wave-body interaction problem for diffraction and radiation. Roll damping was modeled to match experimental roll decay tests.

HISTOGRAM OF PARAMETRIC ROLL

Fig. 1 shows a histogram using all points of the dataset. As expected (and as demonstrated by a number of sources referenced in the previous section), the shape of the distribution is visually far from normal.

The width of the bins for the histogram in Fig. 1 was calculated with the following formula (Scott, 1979):

$$W = \frac{3.5\sigma}{\sqrt[3]{N_p}} \tag{1}$$

Where σ is standard deviation and N_p is the number of available data points.

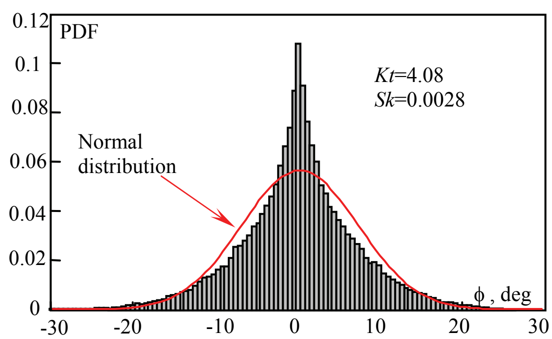


Fig. 1: Histogram of parametric roll shown along with normal distribution

The statistical estimate of the kurtosis was *Kt*=4.08. This is above the kurtosis of a normal distribution with an excess of kurtosis equal to 1.08. The observed distribution is leptokurtic, meaning that the distribution has a peak that is sharper than normal, as can be clearly seen in Fig. 1.

It also means that the tail is expected to be thicker that normal distribution. Indeed, this can be seen by zooming into the histogram, as shown in Fig. 2.

In order to find the reason for the observed shape, time histories of several records from the data set can be examined, as in Fig. 3. As the calculations were performed in long-crested head waves, almost no roll oscillations occurred outside of the parametric resonance episodes. As a result, a significant time is spent with roll close to 0, which is reflected in the very sharp peak.

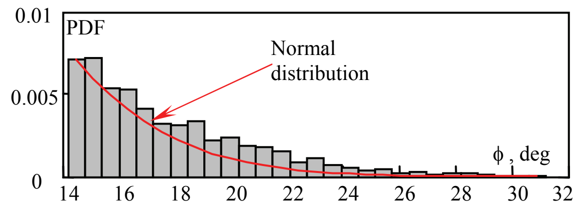


Fig. 2: Detail of histogram revealing thick tail

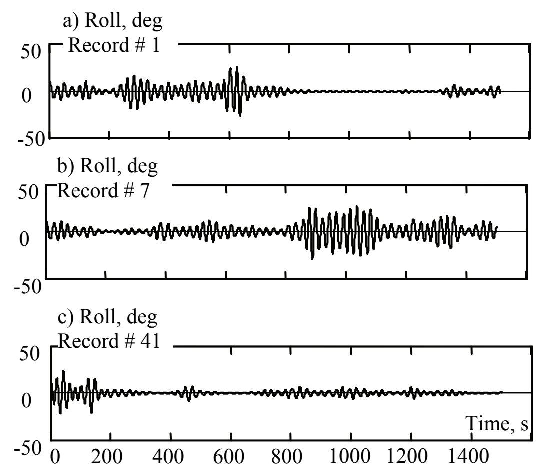


Fig. 3: Sample records of parametric roll in head waves

The thick tails appear to be the result of the extended periods of the large amplitude responses typical for parametric roll episodes. Since the purpose of statistical processing on parametric roll data is to characterize and predict large roll angles, the emphasis of further study should be on the kurtosis.

DISTRIBUTIONS WITH GIVEN KURTOSIS

One of the simplest formulas providing a given kurtosis is the Gram-Charlier distribution, which is expressed as:

$$f(z) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \times \left(1 - \frac{Sk}{3!}(3z - z^3) + \frac{Ex}{4!}(z^4 - 6z^2 + 3)\right) \tag{2}$$

Here, $z = (x - m)/\sigma$, σ is a standard deviation, *Sk* is skewness, and *Kt* is kurtosis, and *Ex* is excess kurtosis. The latter two terms are expressed through the central statistical moments of the third and fourth order:

$$Sk = \frac{M_3}{\sigma^3}; \quad Ex = Kt - 3; \quad Kt = \frac{M_4}{\sigma^4} \tag{3}$$

Formula (2) represents the first several terms of an asymptotically converging series; its convergence with these terms only is not guaranteed. As a result of non-convergence, the PDF may experience oscillations. Such a situation was reported by Belenky and Weems (2008) while attempting a fit of the Gram-Charlier distribution to nonlinear roll motion data.

Another candidate to consider is the Pearson family of distributions, which also allows models a distribution with the given kurtosis but does not have the problem with convergence since it is a solution of a differential equation, not a series expansion. Different types of distributions in the Pearson family correspond to different combination of excess of kurtosis and skewness. For the leptokurtic distribution with relatively small skewness, the Pearson type IV distribution may be applicable. It is presented as:

$$f(x) = N\left(1 + \left(\frac{x-\lambda}{\alpha}\right)^2\right)^{-q} \times \exp\left(-\nu \cdot \arctan\left(\frac{x-\lambda}{\alpha}\right)\right) \tag{4}$$

N is a normalization coefficient, where

$$N = \frac{\Gamma(q)}{\sqrt{\pi}\cdot\alpha\cdot\Gamma(q-0.5)}\left(\left|\frac{\Gamma(q-0.5i\cdot\nu)}{\Gamma(q)}\right|\right)^2 \tag{5}$$

$$\lambda = m - \frac{(r-2)\cdot Sk\cdot\sigma}{4} \tag{6}$$

$$\nu = -\frac{r\cdot(r-2)\cdot Sk}{\sqrt{16(r-1) - Sk^2\cdot(r-2)^2}} \tag{7}$$

$$\alpha = \frac{1}{4}\sigma\sqrt{16(r-1) - Sk^2\cdot(r-2)^2} \tag{8}$$

$$r = \frac{6(Kt - Sk - 1)}{2\cdot Kt - 3\cdot Sk - 6} \tag{9}$$

$$q = 0.5r + 1 \tag{10}$$

Here $i = \sqrt{-1}$.

As the value of Skewness is very small, a Pearson type VII distribution can probably also be considered:

$$f(z) = \frac{1}{\sigma}\frac{\Gamma(q)}{\sqrt{\pi}\cdot\alpha\cdot\Gamma(q-0.5)}\left(1 + \left(\frac{z}{a}\right)^2\right)^{-p} \tag{11}$$

$$a = \sqrt{2 + \frac{6}{Ex}} \quad ; \quad p = \frac{5}{2} + \frac{3}{Ex} \tag{12}$$

Where $z = (x - m)/\sigma$.

All three distributions are shown in Fig. 4, with the normal distribution included for reference. As expected, the Pearson type IV and type VII distributions are practically identical.

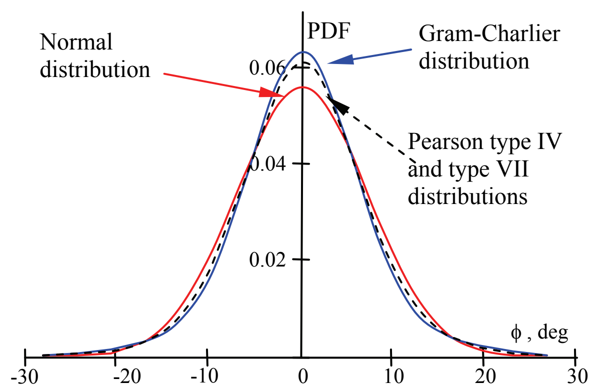
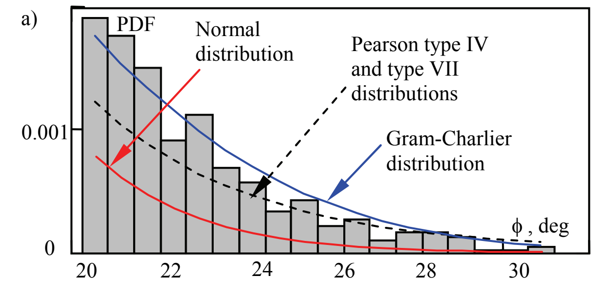


Fig. 4: Theoretical distributions

Despite the fact that the kurtosis is almost exactly the same for the Gram-Charlier and Pearson distributions, the former has a slightly larger peak value than the latter. On the other hand, the tail of Pearson distribution becomes thicker around 28.5 deg.

Fig. 5 plots the tails of the distributions over the corresponding part of the histogram. Visually, Gram-Charlier seems to do a better job than Pearson when considering the range of angles for 20-32 deg (Fig. 5a), but further zooming-in (Fig. 5b) does not support this impression.

Neither the Gram-Charlier nor the Pearson distributions have the most characteristic feature of the histogram of parametric roll shown in Fig. 1, which is the very sharp peak around 0.0 caused by prolonged episodes of small-amplitude roll.



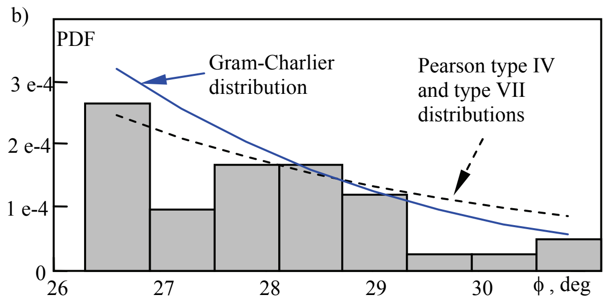


Fig 5: Zoomed-in tails of distributions

MODELING DISTRIBUTION WITH MOVING AVERAGE

The histogram of parametric roll in Fig. 1 looks almost smooth, as numerical simulations of ship motions produce a very large number of data points. There are several reasons for this. The first is that the time step for integration must be small in order to ensure the numerical stability of the hydrodynamic solver. The second is that the number of records should be sufficient to overcome "practical non-ergodicity," while the duration of each record should be longer than the time for the autocorrelation to decay. This time may be not small, because parametric roll is a narrow-band process.

Since a large number of points are needed for the correct statistical characterization of the process, sufficient data may be available to obtain the PDF directly. The idea is to apply the smoothing technique that is used for spectral processing. The simplest method is a "boxcar" or moving average:

$$H_i = \frac{1}{n}\sum_{j=1}^{n} H0_{j+i} \tag{13}$$

Here $H0$ is an original histogram shown in Fig. 1 and n is a number of neighbor bins used for averaging. To preserve the sharp peak, a number of bins around the peak are not averaged. The averaged points are then used with linear interpolation to create a continuous piecewise linear function.

To qualify as a PDF, this function must be corrected in order to satisfy normalization conditions as well as an equality of mean value and variance (Belenky and Weems 2008). Since the mean value is essentially zero, the approximation of the PDF can be expressed as:

$$f(\phi) = K_V^{-1} K_n f_{PWL}(K_V^{-1} \cdot \phi)) \tag{14}$$

f_{PWL} is the piecewise linear function and K_n is the normalization factor, which is computed as:

$$K_n = \left(\int_{\phi_B}^{\phi_E} f_{PWL}(\phi) d\phi \right)^{-1} \tag{15}$$

ϕ_B and ϕ_E are values corresponding to the first and last points of the histogram array H.

K_V is the calibration factor for variance:

$$K_V = \sqrt{\frac{V^*}{V_{PWL}}} \tag{16}$$

V_{PWL} is the variance of the basis function before the normalization:

$$V_{PWL} = K_n \int_{\phi_B}^{\phi_E} \phi^2 \cdot f_{PWL}(\phi) d\phi \tag{17}$$

The approximate PDF is shown in Fig. 6. In this figure, n=7 and the number of points-to-keep around the peak is 5.

The curve in Fig. 6 looks smooth and visually fits the histogram very well. The value of kurtosis is 4.02, which is close enough to the statistical estimate.

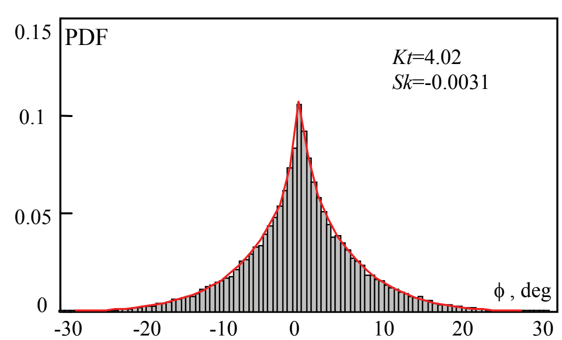


Fig. 6: Moving average approximation of PDF

GOODNESS OF FIT

Three distributions were considered in the two previous sections, all of which were capable of modeling the kurtosis. However, only the "moving average" approximation is capable of retaining the sharp peak, which is the most striking feature of the parametric roll histogram in Fig. 1.

Goodness-of-fit tests are essentially mathematical tools that check if the histogram rejects a theoretical PDF (or just a curve that has the properties of a PDF). Strictly speaking, passing a goodness-of-fit test cannot be considered as formal proof that the random variable or stochastic process has the tested distribution; however, if the goodness-of-fit test is negative, the tested distribution is most probably not applicable for the case. The Pearson chi-square test is one of the most frequently used. Its main advantage that it is adjustable for statistically estimated parameters of the distribution.

The application of the Pearson chi-square goodness-of-fit test for a stochastic process has certain specifics. The chi-square distribution utilized in the test is derived with the assumption that all data points are independent. In the case of the present data, the values in neighbor bins may be dependent, as the time step is relatively small and sequential points of the time history are often counted in the same bin.

As a result, the value of chi-square may be too large for the total number of points available, since the dependent data do not carry as much information as independent data. To avoid this dependency between bins, the sampling time step needs to be increased.

Theoretically, if the interval between the process data points equals or exceeds the time of decay of the autocorrelation function, the histogram bins are independent because the data points are independent. Such a strict approach, however, may be not necessary, and may even be impractical, for a process like parametric roll. It seems sufficient to increase the sampling time step to one-half of the mean zero-crossing period (if Formula (1) is used for a bin's width), at which point the subsequent points, while still dependent, will hit bins that are far from each other and the influence of the dependence will not be felt.

To illustrate this practice, consider 50 records of wave elevations of 30 minutes each (Bretschneider spectrum, significant wave height Hs=11.5 m, modal period Tm=16.4 s and mean zero-crossing period Tz=11.6 s).

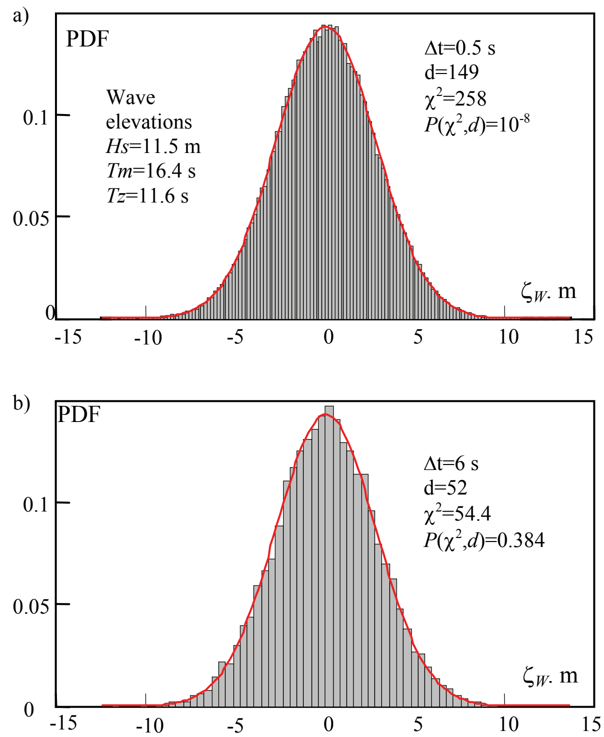


Fig. 7: Test example – histogram of wave elevation

Fig. 7a shows a histogram and results of the Pearson chi-square goodness-of-fit test on all points of all records with time step of 0.5. Despite the fact that the distribution is known to be normal, the test returns a negative result. Increasing the time step up to 6 seconds reverses the result of the test. The time for the decay of the autocorrelation function is about 40 seconds.

The histogram for the goodness-of-fit test for parametric roll is shown in Fig. 8, with the results presented in Table 1.

As can be seen from Table 1, all distributions except the "moving average" model were rejected by the statistical data. The reason is obvious: the sharp peak caused by small-amplitude episodes was captured only in the "moving average" distribution.

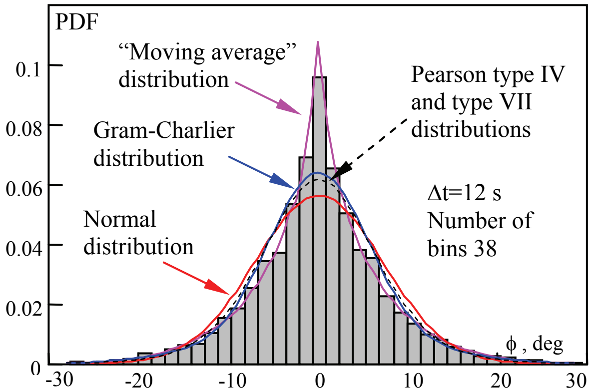


Fig. 8: Goodness-of-fit test for parametric roll

Table 1: Goodness-of-fit test for parametric roll

Distribution	***d***	χ^2	$P(\chi^2,d)$
Normal	36	529	0
Gram-Charlier	34	194	0
Pearson	34	259	0
Moving average	37	39	0.375

APPROXIMATION OF TAILS

The "moving average" distribution is, in fact, a truncated distribution, as it is limited to the first and the last terms of an array resulting from the boxcar procedure. To turn it into a "full-service" distribution, the tails should be modeled as well. An attempt to approximate the tails with a hyperbolic function is described in Belenky and Weems (2008). This is a formidable task, as it is a statistical extrapolation of the extreme response of a highly nonlinear system.

The difficulties associated with the statistical extrapolation of the nonlinear ship roll response come from the fact that nonlinearity only affects large-amplitude motions. Any statistical estimates taken over the entire data set will be dominated by small motions and are therefore equivalent to some sort of implicit linearization.

The peak-over-threshold (POT) method has the potential to solve these types of problems. While well-known in other fields, this method has only recently been closely studied for applications related to ship dynamics. To address issues related with narrow banded processes and both-sides exceedances, the method has been extended for the envelope of the response – EPOT, or envelope peak over threshold (Campbell and Belenky 2010a, 2010b; Belenky and Campbell 2011).

The general scheme for the statistical extrapolation is illustrated in Fig. 9. The problem of extrapolation is separated into two sub-problems: rare and non-rare.

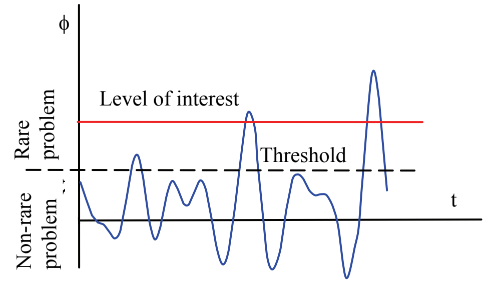


Fig. 9: Scheme of statistical extrapolation with POT/EPOT method

The non-rare sub-problem is focused on the statistical characterization of the upcrossing through a given intermediate level separating small-amplitude and large-amplitude motions. The rare problem can be considered as an extreme value problem for only the data exceeding the threshold.

The result is presented in a form of the upcrossing rate of the level of interest a_2,

$$\xi_2(a_2) = -\frac{1}{T_W}\ln\left(\exp(-\xi_1 T_W) + \left(1-\exp(-\xi_1 T_W)\right)F_{EV}(a_2 \mid a_1, T_W)\right) \tag{18}$$

where a_1 is the threshold, ξ_1 is the upcrossing rate for the threshold, and F_{EV} is a cumulative extreme value distribution (Weibull) that has been fitted using a time window of duration T_W.

Since ξ_2 is the rate of upcrossing through the level a_2, it can also be expressed as:

$$\xi_2(a_2) = f(\phi = a_2)\int_0^\infty \dot\phi\, f(\dot\phi)d\dot\phi \tag{19}$$

Considering the level a_2 as a variable, one can write:

$$f(\phi) = \frac{\xi_2(\phi)}{\int_0^\infty \dot\phi\, f(\dot\phi)d\dot\phi} \tag{20}$$

The denominator of Formula (20) is a constant that can be calculated using the "moving average" approximation distribution of the roll rates shown in Fig. 10. As the denominator is an integral the truncated distribution is sufficient.

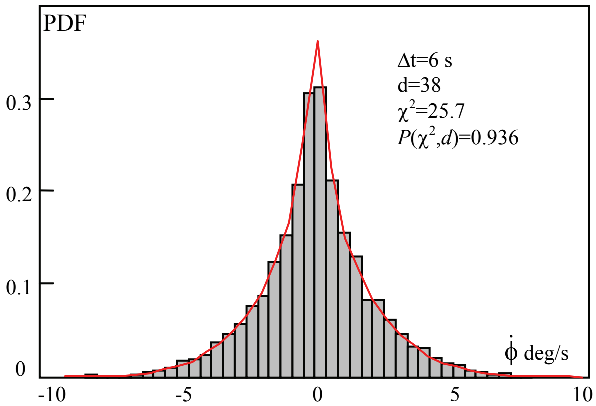


Fig. 10: "Moving average" distribution for roll rate with parametric resonance present

The POT/EPOT method, however, has a limitation of applicability. As the properties of dynamical system change around the maximum of the roll restoring (GZ) curve, the application of POT/EPOT beyond the maximum of the GZ curve may be problematic (Campbell and Belenky 2010a). This effectively limits the length of the tail that can be described with this method. However, for the purposes of the statistical prediction of parametric roll using numerical simulation, this seems to be sufficient.

SUMMARY

The distribution of the roll angle response for a ship susceptible to parametric roll resonance in head long-crested seas is leptokurtic, with a sharp peaks and thick tails. The reason for this distribution shape is a combination of small- and large-amplitude motion episodes that are typical for parametric roll.

Gram-Charlier and Pearson types IV and VII distributions provide a visually reasonable approximation for the tails, but are not capable of reproducing the sharp peak.

As numerical simulations provide a large number of data points, the distribution can be approximated with a "moving average" and linear interpolation.

It was shown that the "moving average" distribution is not rejected by the statistical data using a Pearson chi-square goodness-of-fit test. In order to use this test on dependent data points from multiple records, it is enough to keep about half of a mean zero-crossing period between points. This result may be especially important for the analysis of parametric roll, where the time for the decay of autocorrelation function is usually large and a requirement for independent data points may be impractical.

The POT/EPOT method can be used to approximate a t`ail for the "moving average" distribution.

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