Computing Hydrodynamic Forces and Moments on a Vessel without Bernoulli's Equation

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ABSTRACT

Traditionally the hydrodynamic force on a ship's hull is obtained by integrating the pressure over the hull, using Bernoulli's equation to compute the pressures. Due the need to evaluate Φ_t, Φ_x, Φ_y, Φ_z at every instant in time, this becomes a computational challenge when one wishes to know the hydrodynamic forces (and moments) on the instantaneous wetted surface of a vessel in extreme seas. A methodology that converts the integration of the pressure over the hull surface into an impulse, the time derivative of several integrals of the velocity potential over the surface of the vessel and possibly the free surface near the vessel is introduced.

KEYWORDS

Hydrodynamic forces; Momnetum theory; Hydrodynamic impulse

INTRODUCTION

Sclavounos, Telste and Reed (Sclavounos, 2011; Sclavounos, *et al.*, 2011) have developed a non-linear slender-body model for the treatment of the potential flow problem governing the responses of a vessel in steep random waves. Boundary value problems have been derived for the disturbance radiation and diffraction velocity potentials relative to the ship-fixed coordinate system. The evaluation of the sectional force (and moment) distributions based on the solution of these potential flow sectional boundary value problems is the subject of the present paper. A sectional force method treats as unknown the sectional force distribution along the ship length as opposed to the local pressure, which is natural within a slender-body framework. Combined with additive viscous models, these sectional force models lead to the evaluation of the integrated forces and moments which are input to the vessel nonlinear equations of motion.

BACKGROUND

Traditionally, the derivation of correct sectional force distributions has played a central role in slender-body theory of aerodynamics and hydrodynamics. The direct application of Bernoulli's equation is complicated by a number of facts. The first is the need to evaluate gradients of the velocity potential, which may be a delicate computational task within a panel method. The second is the proper treatment of the longitudinal gradients of the ambient and disturbance potentials, which may not be possible to ignore in light of the slenderness approximations. A third fact which arises in connection with the present nonlinear time-domain slender-body theory is the proper interpretation of time derivatives with respect to the ship-fixed coordinate system and their careful treatment in the vicinity of the free-surface ship-hull intersection.

These complications with the direct application of Bernoulli's equation within a slender-body theory are mitigated if the integrated sectional forces are instead evaluated by the proper application of the momentum conservation theorem. This approach has several merits that have been taken into account in the development of strip theory and subsequent linear and nonlinear slender-body theories. Drawing upon the work of Lighthill (1960) and Newman & Wu (1973) on the swimming of slender fish, expressions can be derived for the sectional force distributions which are simple functions of the sectional integrals of the velocity potential. This important result circumvents in an elegant and robust manner the need to interpret the longitudinal convective terms in Bernoulli's equation. Moreover, the presence of a sectional integral of the velocity potential in the force expression suggest that this is the fundamental quantity needed for the evaluation of the sectional and total forces, as opposed to the local values of the pressure or velocity potential. This in turn may lead to simple—or even analytical—expressions for the sectional force distributions within a slender-body framework in a number of settings. Finally, this sectional force formulation allows for a simple and robust interpretation of time derivatives when the sectional wetted surface is time dependent as the vessel sections move in and out of the free surface.

THE BOUNDARY VALUE PROBLEM AND ITS DECOMPOSITION

Let us assume as an earth-fixed reference a right-handed coordinate system (X,Y,Z) and a ship-fixed right-handed coordinate system (x,y,z) centered at an arbitrary point B with the xy-plane parallel to the calm water surface $Z=0$ when the ship is at rest (Figure 1). The ship position in space is completely defined by the rectilinear displacement vector $\boldsymbol{\Xi}_B(t) = \xi_1(t)\boldsymbol{i} + \xi_2(t)\boldsymbol{j} + \xi_3(t)\boldsymbol{k}$ from the origin of the earth-fixed coordinate system to the origin of the ship-fixed coordinate system and the Euler angles defined in the order $[\xi_6(t), \xi_5(t), \xi_4(t)]$.

The free surface is assumed to be a single-valued function of the horizontal coordinates X and Y. Surface tension is negligible. The fluid is assumed to be homogeneous, incompressible, and frictionless. The fluid flow is assumed to be irrotational. These conditions are sufficient to guarantee the existence of a velocity potential.

The Total Velocity Potential

In the fluid surrounding the ship, the total velocity potential is Φ. It satisfies the Laplace equation

$$\nabla^2\Phi = 0$$

within the fluid domain bounded by the free-surface $Z = \zeta(X,Y,t)$ and the hull of the ship. The total potential satisfies at least a linear free-surface boundary condition on $Z = \zeta(X,Y,t)$.[1]

Hull Boundary Condition for the Total Potential

Since the hull boundary condition is derived with vectors defined to be independent of frames of reference or coordinate systems, the appropriate boundary condition expressed in terms of either the earth-fixed or ship-fixed frame of reference is obtained from the components of vector equations.

Velocity of Points on the Hull

To obtain the hull boundary condition satisfied by Φ, we first consider a point fixed on the hull (fixed in the ship-fixed frame of reference). It has ship-fixed coordinates x, y, z and earth-fixed coordinates X, Y, Z. The vectors $\boldsymbol{x}$ and $\boldsymbol{X}$ from the origins of the ship-fixed and earth-fixed coordinate systems to the point, respectively, satisfy the equations

$$\begin{aligned} \boldsymbol{X} &= X\boldsymbol{i} + Y\boldsymbol{j} + Z\boldsymbol{k} \\ \boldsymbol{x} &= x\bar{\boldsymbol{i}} + y\bar{\boldsymbol{j}} + z\bar{\boldsymbol{k}} \\ \boldsymbol{x} &= \boldsymbol{X} - \boldsymbol{\Xi}_B(t) \end{aligned}$$

where $\boldsymbol{\Xi}_B(t) = \xi_1\boldsymbol{i} + \xi_2\boldsymbol{j} + \xi_3\boldsymbol{k}$ is the vector from the origin of the earth-fixed coordinate system to the origin of the ship-fixed coordinate system. Since the point is fixed on the hull, x, y, and z are independent of time. Consequently, the ship-fixed time derivative of $\boldsymbol{x}$ vanishes. We have

$$\mathbf{0} = \frac{d^*\boldsymbol{x}}{dt} = \frac{d\boldsymbol{X}}{dt} - \frac{d\boldsymbol{\Xi}_B}{dt} - \boldsymbol{\Omega}\times\boldsymbol{x}$$

[1] As the free-surface boundary condition is not used in the development of the momentum theory for the force, the specific free-surface boundary condition chosen is not important.

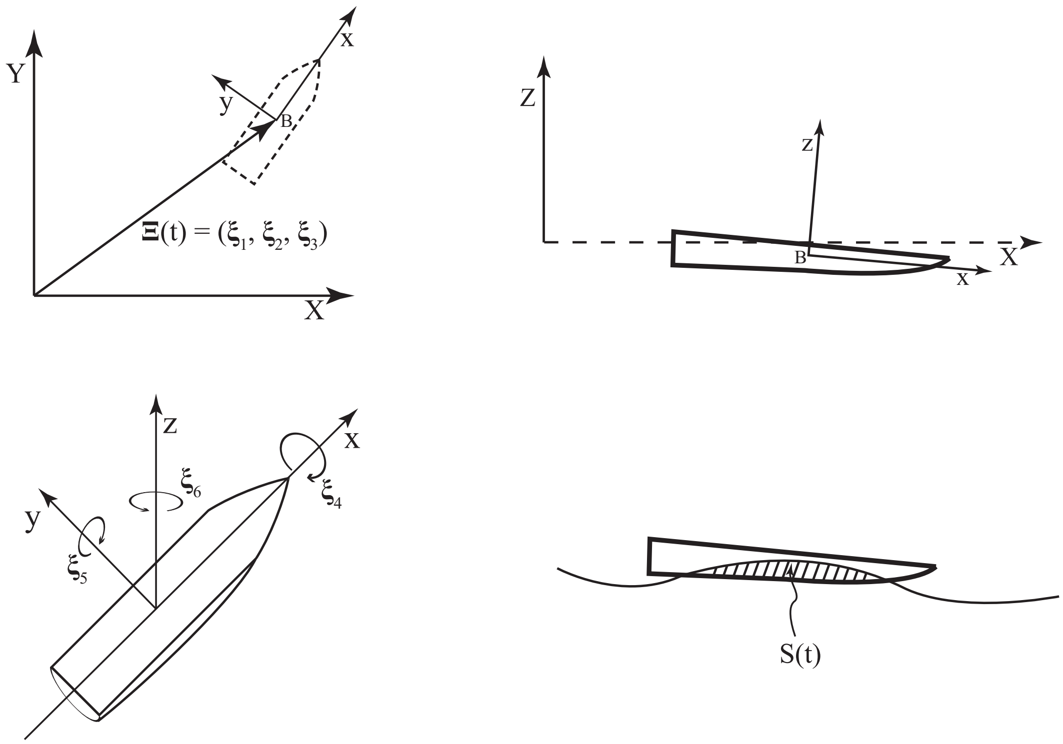

Fig. 1 Coordinate systems for the nonlinear ship response problem.

where d^*/dt and d/dt operating on a vector obtain the ship-fixed and earth-fixed time derivatives of the vector, respectively. The velocity $\boldsymbol{v}_{SHIP}$ of the point on the hull is then

$$\boldsymbol{v}_{SHIP} = \frac{d\boldsymbol{X}}{dt} = \frac{d\boldsymbol{\Xi}_B}{dt} + \boldsymbol{\Omega} \times \boldsymbol{x}. \qquad (1)$$

If the point slides along the hull surface, it is not fixed in the ship-fixed frame of reference and $d^*\boldsymbol{x}/dt \neq \mathbf{0}$. However, it is true that $\boldsymbol{n} \cdot d^*\boldsymbol{x}/dt = 0$ so that

$$\boldsymbol{n} \cdot \frac{d\boldsymbol{X}}{dt} = \boldsymbol{n} \cdot \left(\frac{d\boldsymbol{\Xi}_B}{dt} + \boldsymbol{\Omega} \times \boldsymbol{x} \right) \boldsymbol{n} \cdot \boldsymbol{v}_{SHIP}$$

where $\boldsymbol{v}_{SHIP}$ is the velocity of a fixed point on the hull coinciding with the position of the sliding point at time t.

Equation for the Hull Surface

The hull surface $S_B(t)$ is rigid and therefore independent of time in the ship-fixed frame of reference. Points on the surface are those points whose ship-fixed coordinates x, y, z satisfy a mathematical equation of the form

$$h(x,y,z) = 0\,.$$

The unit normal $\boldsymbol{n}$ on the hull surface is defined by the equation

$$\boldsymbol{n} = \frac{\nabla h}{|\nabla h|}\,.$$

The gradient points in the direction of maximum increase of h. For that reason it is stipulated that $h(x,y,z) > 0$ for points inside the hull with ship-fixed coordinates x, y, z and $h(x,y,z) < 0$ for points outside the hull. Then $\boldsymbol{n}$ is guaranteed to point into the hull.

Hull Boundary Condition

To obtain the hull boundary condition, we now consider an arbitrary point with ship-fixed coordinates x, y, z and earth-fixed coordinates X, Y, Z. The point moves and traces out a smooth trajectory so that both the ship-fixed and earth-fixed coordinates are functions of time. The derivative of h following the point is

$$\begin{aligned}\frac{Dh}{Dt} &= \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial z}\frac{dz}{dt} \\ &= \nabla h \cdot \frac{d^* \boldsymbol{x}}{dt} \\ &= \nabla h \cdot \left(\frac{d\boldsymbol{X}}{dt} - \frac{d\boldsymbol{\Xi}_B}{dt} - \boldsymbol{\Omega} \times \boldsymbol{x} \right) .\end{aligned}$$

If the point is a fluid particle sliding along the surface of the hull, then $Dh/Dt = 0$, $d\boldsymbol{X}/dt = \nabla\Phi$, and

$$0 = \boldsymbol{n} \cdot \left(\nabla\Phi - \frac{d\boldsymbol{\Xi}_B}{dt} - \boldsymbol{\Omega} \times \boldsymbol{x} \right) .$$

The hull surface boundary condition requires that the normal velocity of a fluid particle on the hull surface match the normal velocity of the hull:

$$\boldsymbol{n} \cdot \nabla\Phi = \boldsymbol{n} \cdot \left(\frac{d\boldsymbol{\Xi}_B}{dt} + \boldsymbol{\Omega} \times \boldsymbol{x} \right) \boldsymbol{n} \cdot \boldsymbol{v}_{SHIP}$$

where $\boldsymbol{v}_{SHIP}$ is given by (1).

Incident Wave Potential

In the absence of a body, the velocity potential would have been the ambient velocity potential ϕ_I, which satisfies the Laplace equation

$$\nabla^2 \phi_I = 0$$

in the fluid below the free-surface elevation $Z = \zeta_I(X,Y,t)$.

The ambient wave velocity potential ϕ_I is assumed to satisfy the same free-surface boundary condition as the total velocity potential, but on $Z = \zeta_I(X,Y,t)$. It is also assumed that the incident wave elevation differs little from the total wave elevation except possibly near the vessel.

The Disturbance Velocity Potential

When $\zeta_I < \zeta$, it is assumed that ϕ_I can be analytically continued above $Z = \zeta_I$ to define a continuation everywhere outside the hull in the fluid below the free-surface elevation $Z\zeta$. Then a disturbance potential ϕ_D is defined everywhere in this domain according to the equation

$$\phi_D = \Phi - \phi_I .$$

The difference in wave elevation between the total wave elevation around the hull and the ambient wave potential that would have existed in the absence of the ship is ζ_D. It obviously satisfies the equation

$$\zeta_D = \zeta - \zeta_I .$$

The free-surfce boundary condition for the disturbance potential potential is derived from that of the total velocity potential, substituting $\phi_I + \phi_D$ and $\zeta_I + \zeta_D$ for Φ and ζ in the total velocity potential free-surface boundary condition and linearizing in ϕ_D and ζ_D.

Hull Boundary Condition for the Disturbance Potential

Using the assumed decomposition of the total potential as the sum of the incident wave potential and a disturbance potential, we obtain the equation

$$\boldsymbol{n} \cdot \nabla\phi_D \boldsymbol{n} \cdot \boldsymbol{v}_{SHIP} - \boldsymbol{n} \cdot \nabla\phi_I .$$

where $\boldsymbol{v}_{SHIP}$ is given by (1).

Boundary Condition at Infinity for the Disturbance Potential

At infinity, the velocity due to the disturbance velocity potential approaches zero, and the free-surface waves generated by the interaction of the ship with the ambient waves radiate outward. This is the radiation boundary condition.

THE FLUID FORCE ON THE VESSEL

The purely three-dimensional case of a vessel oscillating in six degrees of freedom in steep ambient waves is considered. The most general fully nonlinear problem is formulated first leading to the treatment of special cases. The sectional force is evaluated first relative to the inertial frame and next relative to the ship-fixed coordinate system.

Figure 2 illustrates a 3D vessel undergoing rectilinear and rotational displacements in steep ambient waves. The fully nonlinear free-surface elevation $\zeta(t)$ is the sum of the ambient wave elevation and the disturbance caused by the vessel displacement and the corresponding total free surface

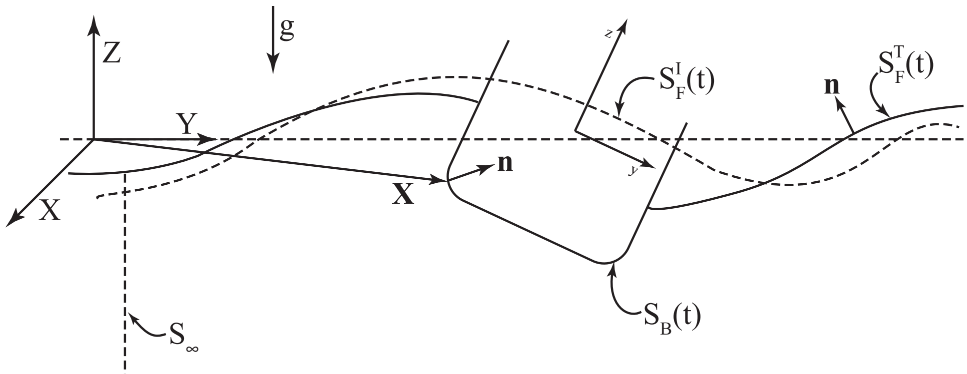

Fig. 2 Coordinate system for vessel undergoing rectilinear and rotational displacement in steep ambient waves

is denoted by $S_F^T(t)$. The nonlinear wave elevation of the ambient wave alone is $\zeta_I(t)$ and the corresponding free-surface elevation is denoted by $S_F^I(t)$. The difference between the two free-surface elevations is assumed to be finite. Yet, this difference is expected to be small, except perhaps near the waterline. This assumption is essential for the derivation of an approximate form of the three-dimensional force acting on the vessel using the momentum theorem developed below. The fluid in the volume V bounded by the wetted surface S_B^T of the hull, the free surface S_F^T, and a control surface S_∞^T is considered:

$$S^T = S_B^T + S_F^T + S_\infty^T.$$

The control surface is fixed with respect to the earth-fixed coordinate system. It is also bounded until the end when the surface is moved to infinity in all directions. The rate of change of the fluid momentum in the volume is

$$\boldsymbol{F}^{\text{fluid}} = \rho \frac{d}{dt} \iiint_V dV\, \nabla\Phi = \rho \frac{d}{dt} \oiint_{S^T} dS\, \Phi \boldsymbol{n} \quad (2)$$

where Gauss' theorem has been used to convert the volume integral to a surface integral. (Here, partial derivatives with respect to time are earth-fixed where the earth-fixed coordinates X, Y, Z of a point in space are fixed.) According to the transport theorem, the rate of change of momentum is also given by the equation

$$\boldsymbol{F}^{\text{fluid}} = \rho \iiint_V dV\, \nabla \frac{\partial \Phi}{\partial t} + \rho \oiint_{S^T} dS\, U_n \nabla\Phi$$
$$= \rho \oiint_{S^T} dS\, \frac{\partial \Phi}{\partial t} \boldsymbol{n} + \rho \oiint_{S^T} dS\, U_n \nabla\Phi$$

where $U_n = \boldsymbol{n} \cdot \boldsymbol{U}$ is the outward normal component of the velocity $\boldsymbol{U}$ of the surface S^T. Putting these results together, we obtain the equation

$$\begin{aligned}
&\rho \frac{d}{dt} \oiint_{S^T} dS\, \Phi \boldsymbol{n} \\
&= \rho \oiint_{S^T} dS\, \frac{\partial \Phi}{\partial t} \boldsymbol{n} + \rho \oiint_{S^T} dS\, U_n \nabla\Phi \\
&= \rho \oiint_{S^T} dS \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right) \boldsymbol{n} \\
&\quad + \rho \oiint_{S^T} dS \Big(U_n \nabla\Phi \\
&\qquad - \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \boldsymbol{n} \Big) \qquad (3) \\
&= \rho \oiint_{S^T} dS \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right) \boldsymbol{n} \\
&\quad + \rho \oiint_{S^T} dS \left(U_n - \frac{\partial \Phi}{\partial n} \right) \nabla\Phi \\
&= \rho \oiint_{S^T} dS \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} \nabla\Phi \cdot \nabla\Phi \right) \boldsymbol{n} \\
&\quad - \rho \iint_{S_\infty^T} dS \frac{\partial \Phi}{\partial n} \nabla\Phi .
\end{aligned}$$

The third equality is obtained by using Newman's identity

$$\oiint_S dS \left[\nabla\varphi\frac{\partial\varphi}{\partial n} - \frac{1}{2}\nabla\varphi\cdot\nabla\varphi\boldsymbol{n}\right] = 0$$

which holds for any velocity potential φ within a volume enclosed by a surface S [Newman (1977), p. 134, Eq. 89]. The last equality in (3) is obtained from the equations $U_n = 0$ on S_∞^T and $U_n = \partial\Phi/\partial n$ on S_F^T and S_B^T. The total fluid force $\boldsymbol{F}_{\mathrm{TOT}}$ acting on the body is the integral of $p\boldsymbol{n}$ over the wetted surface of the hull. Thus we obtain the equation

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{TOT}} = &-\rho\iint_{S_B^T} dS\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gZ\right)\boldsymbol{n} \\
= &-\rho\frac{d}{dt}\oiint_{S^T} dS\,\Phi\boldsymbol{n} - \rho\iint_{S_\infty^T} dS\frac{\partial\Phi}{\partial n}\nabla\Phi \\
&-\rho g\iint_{S_B^T+S_F^T} dS\,Z\boldsymbol{n} \\
&+\rho\iint_{S_\infty^T} dS\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)\boldsymbol{n}
\end{aligned} \tag{4}$$

for the total fluid force acting on the body. The fact that the pressure vanishes on S_F^T has been used to obtain this equation.

Equation (4) is an important intermediate result which was derived without invoking any approximations. It accomplishes one of the objectives of the momentum formulation, namely to reduce the definition of the force by pressure integration into integrals that are much easier to evaluate or further reduce as indicated below. The superscript T has been used to indicate surfaces for the total nonlinear problem.

The fluid that would have existed inside the volume bounded by the ambient wave free surface S_F^I and the control surface S_∞^I, if the ship had not disturbed the water, is now considered. The surface S_∞^I is slightly different from S_∞^T only due to the difference between the ambient wave elevation $\zeta_I(t)$ and the total nonlinear wave elevation $\zeta(t)$. The total bounding surface is S^I where

$$S^I = S_F^I + S_\infty^I\,.$$

Just as was done for the fluid in the volume V outside the hull below the surface S_F^T, one can consider the rate of change of the fluid momentum inside the volume bounded by S^I. It can be obtained from eqs. (2)–(3) by letting the hull shrink to infinitesimal size. The integrals over S_B^T then vanish, S_F^T becomes S_F^I and S_∞^T becomes S_∞^I. The force acting on the vanishingly small ship is zero and is given by either side of the equation

$$\begin{aligned}
\boldsymbol{0} = &-\rho\frac{d}{dt}\oiint_{S^I} dS\,\phi_I\boldsymbol{n} - \rho\iint_{S_\infty^I} dS\frac{\partial\phi_I}{\partial n}\nabla\phi_I \\
&-\rho g\iint_{S_F^I} dS\,Z\boldsymbol{n} \\
&+\rho\iint_{S_\infty^I} dS\left(\frac{\partial\phi_I}{\partial t} + \frac{1}{2}\nabla\phi_I\cdot\nabla\phi_I\right)\boldsymbol{n}\,.
\end{aligned} \tag{5}$$

Equation (5) is subtracted from (4) to obtain the equation

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{TOT}} = &-\rho\iint_{S_B^T} dS\left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gZ\right)\boldsymbol{n} \\
= &-\rho\frac{d}{dt}\iint_{S_B^T} dS\,\Phi\boldsymbol{n} - \rho g\iint_{S_B^T} dS\,Z\boldsymbol{n} \\
&-\rho\frac{d}{dt}\left[\iint_{S_F^T} dS\,\Phi\boldsymbol{n} - \iint_{S_F^I} dS\,\phi_I\boldsymbol{n}\right] \\
&-\rho g\left[\iint_{S_F^T} dS\,Z\boldsymbol{n} - \iint_{S_F^I} dS\,Z\boldsymbol{n}\right] \\
&-\rho\frac{d}{dt}\left[\iint_{S_\infty^T} dS\,\Phi\boldsymbol{n} - \iint_{S_\infty^I} dS\,\phi_I\boldsymbol{n}\right]
\end{aligned}$$

$$
\begin{aligned}
&-\rho\left[\iint_{S_\infty^T} dS\,\frac{\partial\Phi}{\partial n}\nabla\Phi\right.\\
&\qquad\left.-\iint_{S_\infty^I} dS\,\frac{\partial\phi_I}{\partial n}\nabla\phi_I\right]\\
&+\rho\left[\iint_{S_\infty^T} dS\left(\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)\boldsymbol{n}\right.\\
&\qquad-\iint_{S_\infty^I} dS\left(\frac{\partial\phi_I}{\partial t}\right.\\
&\qquad\left.\left.-\;+\frac{1}{2}\nabla\phi_I\cdot\nabla\phi_I\right)\boldsymbol{n}\right].
\end{aligned}
$$

It is argued that the sums of the terms within the last three pairs of square brackets are negligibly small when the control surfaces are moved infinitely far away from the ship. The force $\boldsymbol{F}_{\mathrm{DYN}}$ acting on the body due to the dynamic pressure is

$$
\begin{aligned}
\boldsymbol{F}_{\mathrm{DYN}} &= -\rho\iint_{S_B^T} dS\left(\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)\boldsymbol{n}\\
&\simeq -\rho\frac{d}{dt}\iint_{S_B^T} dS\,\Phi\boldsymbol{n}\\
&\quad-\rho\frac{d}{dt}\left[\iint_{S_F^T} dS\,\Phi\boldsymbol{n}-\iint_{S_E^I} dS\,\phi_I\boldsymbol{n}\right]\\
&\quad-\rho g\left[\iint_{S_F^T} dS\,Z\boldsymbol{n}-\iint_{S_E^I} dS\,Z\boldsymbol{n}\right]\\
&\quad+\rho\frac{d}{dt}\iint_{S_W^I} dS\,\phi_I\boldsymbol{n}+\rho g\iint_{S_W^I} dS\,Z\boldsymbol{n}.
\end{aligned}
$$

Here S_E^I is the portion of S_F^I that is outside the hull and S_W^I is the portion of S_F^I that is inside the hull:

$$S_F^I = S_E^I + S_W^I\,.$$

The functions ϕ_I and Φ are continued analytically about $Z=\zeta_I$ and $Z=\zeta$, respectively, so that the function ϕ_I is defined for $Z\le\zeta$ and Φ is defined for $Z\le\zeta_I$. Then Φ may be expanded about the ambient free-surface elevation $Z\zeta_I$. The dynamic force satisfies the approximation

$$
\begin{aligned}
\boldsymbol{F}_{\mathrm{DYN}} &= -\rho\iint_{S_B^T} dS\left(\frac{\partial\Phi}{\partial t}+\frac{1}{2}\nabla\Phi\cdot\nabla\Phi\right)\boldsymbol{n}\\
&\simeq -\rho\frac{d}{dt}\iint_{S_B^T} dS\,\Phi\boldsymbol{n}-\rho\frac{d}{dt}\iint_{S_E^I} dS\,\phi_D\boldsymbol{n}\\
&\quad-\rho g\iint_{S_E^I} dS\,\zeta_D\boldsymbol{n}+\rho\frac{d}{dt}\iint_{S_W^I} dS\,\phi_I\boldsymbol{n}\\
&\quad+\rho g\iint_{S_W^I} dS\,Z\boldsymbol{n}
\end{aligned}
\tag{6}
$$

where $\boldsymbol{n}$ points into the body on S_B^T and upward ($\boldsymbol{n}\cdot\boldsymbol{k}>0$) on S_E^I and S_W^I.

We now follow the steps taken in considering the rate of change of the momentum in the fluid outside the hull in eqs. (2)–(3). However, this time we consider the rate of change of the momentum of the fluid inside the volume bounded by the surface S_B^I and the ambient free surface S_W^I that would have been the case if the ship had not disturbed the fluid. The surface S_B^I is the part of the hull surface that lies below the ambient free surface $Z=\zeta_F^I(t)$. The bounding surface S^{INT} is now the disjoint sum of the hull surface S_B^I and the nonlinear waterline S_W^I:

$$S^{\mathrm{INT}} = S_B^I + S_W^I\,.$$

In this case, the velocity potential is ϕ_I. Since $U_n=\boldsymbol{n}\cdot\boldsymbol{U}$ and $\partial/\partial n=\boldsymbol{n}\cdot\nabla$, the final result given by (3) is unchanged if $\boldsymbol{n}$ is replaced by $-\boldsymbol{n}$. The normal is chosen to point into the volume enclosed by S^{INT} so that it matches the normal on S_B^T in previous equations. In (3), U_n is the same as $\partial\phi_I/\partial n$ on S_W^I. After rearranging terms, the equation corresponding to (3) is therefore

$$
\begin{aligned}
&\rho\oiint_{S^{\mathrm{INT}}} dS\left(\frac{\partial\phi_I}{\partial t}+\frac{1}{2}\nabla\phi_I\cdot\nabla\phi_I\right)\boldsymbol{n}'\\
&\quad=\rho\iint_{S_B^I} dS\left(\frac{\partial\phi_I}{\partial t}+\frac{1}{2}\nabla\phi_I\cdot\nabla\phi_I\right)\boldsymbol{n}'\\
&\quad-\rho g\iint_{S_W^I} dS\,Z\boldsymbol{n}'=
\end{aligned}
\tag{7}
$$

$$= \rho \frac{d}{dt} \oiint_{S^{\mathrm{INT}}} dS\, \phi_I \boldsymbol{n}'$$
$$+ \rho \iint_{S^{\mathrm{INT}}} dS\, \left(\nabla\phi_I \cdot \boldsymbol{n}' - \boldsymbol{U} \cdot \boldsymbol{n}'\right) \nabla\phi_I$$
$$= \rho \frac{d}{dt} \oiint_{S^{\mathrm{INT}}} dS\, \phi_I \boldsymbol{n}'$$
$$+ \rho \iint_{S_B^I} dS\, \left(\nabla\phi_I \cdot \boldsymbol{n}' - \boldsymbol{U} \cdot \boldsymbol{n}'\right) \nabla\phi_I$$

where $\boldsymbol{n}'$ is an inward normal that points into the body on S_B^I and downward on S_W^I.

We now add (7) to (6) while accounting for the different meaning of $\boldsymbol{n}$ and $\boldsymbol{n}'$ on S_W^I in the two equations. The result is the disturbance force $\boldsymbol{F}_D$ given by the equation

$$\begin{aligned}
\boldsymbol{F}_D \simeq & -\rho \iint_{S_B^I} dS \left(\frac{\partial\phi_D}{\partial t} + \frac{1}{2}\nabla\phi_D \cdot \nabla\phi_D \right. \\
& \left. + \nabla\phi_D \cdot \nabla\phi_I\right) \boldsymbol{n} \\
\simeq & -\rho \frac{d}{dt} \iint_{S_B^I} dS\, \phi_D \boldsymbol{n} - \rho \frac{d}{dt} \iint_{S_E^I} dS\, \phi_D \boldsymbol{n} \\
& - \rho g \iint_{S_E^I} dS\, \zeta_D \boldsymbol{n} \\
& + \rho \iint_{S_B^I} dS \left(\frac{\partial\phi_I}{\partial n} - U_n\right) \nabla\phi_I ,
\end{aligned}$$

which assumes that an integral over S_B^T is approximated well by an integral over S_B^I. This is the part of the dynamic force acting on the body that depends on ϕ_D. The part that depends on ϕ_I but not on ϕ_D is obtained from (7):

$$\begin{aligned}
\boldsymbol{F}_{F\text{-}K} = & -\rho \iint_{S_B^I} dS \left(\frac{\partial\phi_I}{\partial t} + \frac{1}{2}\nabla\phi_I \cdot \nabla\phi_I\right) \boldsymbol{n} \\
& - \rho \frac{d}{dt} \iint_{S_B^I} dS\, \phi_I \boldsymbol{n} + \rho \frac{d}{dt} \iint_{S_W^I} dS\, \phi_I \boldsymbol{n} \\
& - \rho \iint_{S_B^I} dS \left(\frac{\partial\phi_I}{\partial n} - U_n\right) \nabla\phi_I \\
& + \rho g \iint_{S_W^I} dS\, Z \boldsymbol{n}
\end{aligned} \tag{8}$$

where $\boldsymbol{n}$ points into the body on S_B^I and upward on S_W^I. The sum of the nonlinear Froude-Krylov and disturbance forces is

$$\begin{aligned}
\boldsymbol{F} = & -\rho \iint_{S_B} dS \left(\frac{\partial\Phi}{\partial t} + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi\right) \boldsymbol{n} \\
\simeq & -\rho \frac{d}{dt} \iint_{S_B^I} dS\, \phi_D \boldsymbol{n} - \rho \frac{d}{dt} \oiint_{S_B^I + S_W^I} dS\, \phi_I \boldsymbol{n}' \\
& - \rho g \iint_{S_W^I} dS\, Z \boldsymbol{n}' \\
& - \rho \frac{d}{dt} \iint_{S_E^I} dS\, \phi_D \boldsymbol{n} - \rho g \iint_{S_E^I} dS\, \zeta_D \boldsymbol{n}
\end{aligned} \tag{9}$$

where the unit normal $\boldsymbol{n}'$ points into the body on S_B^I and downward on S_W^I.

The total force acting on the vessel may be obtained by adding the force due to the hydrostatic pressure in (9) as shown in the equation

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{TOT}} = & -\rho \iint_{S_B} dS \left(\frac{\partial\Phi}{\partial t}\right. \\
& \left. + \frac{1}{2}\nabla\Phi \cdot \nabla\Phi + gZ\right) \boldsymbol{n} \\
\simeq & -\rho \frac{d}{dt} \iint_{S_B^I} dS\, \phi_D \boldsymbol{n} - \rho \frac{d}{dt} \oiint_{S_B^I + S_W^I} dS\, \phi_I \boldsymbol{n}' \\
& - \rho g \oiint_{S_B^I + S_W^I} dS\, Z \boldsymbol{n}' \\
& - \rho \frac{d}{dt} \iint_{S_E^I} dS\, \phi_D \boldsymbol{n} - \rho g \iint_{S_E^I} dS\, \zeta_D \boldsymbol{n} ,
\end{aligned} \tag{10}$$

where $\boldsymbol{n}'$ points downward on S_W^I and into the body on S_B^I. Three force components may be identified in (10).

Nonlinear Buoyancy Force.

Applying the Gauss divergence theorem to the third term on the right side of (10), we obtain

$$\widetilde{\boldsymbol{F}}_H = -\rho g \oiint_{S_B^I + S_W^I} dS\, Z \boldsymbol{n}' = \rho g \triangledown(t) \boldsymbol{k} \tag{11}$$

where $\boldsymbol{n}'$ points into the enclosed volume. The nonlinear hydrostatic force given by (11) acts in the vertical direction and on the volume of fluid enclosed by the ship wetted surface and the ambient wave surface interior to the vessel. This buoyancy force which results from the application of the momentum theorem differs from the conventional hydrostatic force that acts on the open wetted surface of the body and which in the nonlinear problem may not point in the vertical direction.

Momentum Froude-Krylov Force.

The momentum Froude-Krylov force is the time derivative of the impulse integral involving just the ambient wave potential over the instantaneous ship surface:

$$\widetilde{\boldsymbol{F}}_{\text{F-K}} = -\rho \frac{d}{dt} \oiint_{S_B^I(t)+S_W^I(t)} dS\, \phi_I \boldsymbol{n}' . \tag{12}$$

Again $\boldsymbol{n}'$ points into the enclosed volume. The momentum Froude-Krylov force given by (12) differs from the conventional Froude-Krylov force $\boldsymbol{F}_{\text{F-K}}$ which involves the integral of the hydrodynamic pressure due to the ambient wave over the instantaneous ship wetted surface. Although a different force, expression (12) is simpler to evaluate numerically since it does not involve the time derivative and spatial gradients of the ambient velocity potential under the integral sign.

Momentum radiation and diffraction force.

The momentum disturbance force has a similar form to its Froude-Krylov counterpart and involves the disturbance radiation and diffraction velocity potentials under the integral sign in the definition of the corresponding impulse

$$\widetilde{\boldsymbol{F}}_D = -\rho \frac{d}{dt} \iint_{S_B^I(t)} dS\, (\phi_{\text{RAD}} + \phi_{\text{DIF}}) \boldsymbol{n} . \tag{13}$$

An advantage of (13) relative to the conventional definition of the nonlinear radiation and diffraction forces is that no time derivative and spatial gradients of the disturbance potentials are present under the integral sign in the definition of the disturbance impulse. This is a significant advantage of (13) which may be readily evaluated robustly assuming knowledge of just the values of the disturbance velocity potentials over the instantaneous ship wetted surface.

Interpretation of Momentum Hydrostatic and Froude-Krylov Forces

The momentum formulation derived above decomposes the total ideal fluid force into three components which are interpreted as the Momentum Hydrostatic, Froude-Krylov (F-K) and Disturbance Forces.

There exists an interdependence between the hydrostatic and Froude-Krylov forces, the understanding of which in the nonlinear ship response problem is essential for the study of the vessel stability problem in steep waves. As pointed out by Telste & Belknap (2008) and Belknap & Telste (2008), the nonlinear hydrostatic and Froude-Krylov force may cancel each other out in certain wave conditions, underscoring the significance of the accurate evaluation of these forces and the remaining disturbance forces. The discussion below explains how such a cancellation occurs.

The Momentum F-K Force

The derivation of the momentum hydrostatic and Froude-Krylov forces entailed no approximations in the use of Bernoulli's equation so they are considered exact, given an accurate representation of the kinematics of the ambient wave. The hydrostatic force always points upwards and its magnitude depends on the time dependent displaced volume of the vessel and is given by expression (11).

The nonlinear Froude-Krylov force given by (8) may be reduced further by adding and subtracting an integral over the nonlinear waterplane area of the vessel over the ambient wave free surface internal to the vessel:

$$\begin{aligned} \boldsymbol{F}_{\text{F-K}} &= -\rho \frac{d}{dt} \oiint_{S_B^I(t)+S_W^I(t)} dS\, \phi_I \boldsymbol{n}' + \rho \frac{d}{dt} \iint_{S_W^I(t)} dS\, \phi_I \boldsymbol{n}' \\ &= \widetilde{\boldsymbol{F}}_{\text{F-K}} + \rho \frac{d}{dt} \iint_{S_W^I(t)} dS\, \phi_I \boldsymbol{n}' . \end{aligned} \tag{14}$$

Here $\boldsymbol{n}'$ points downward on S_W^I and into the body on S_B. In (14) the first integral is over a surface enclosing the time dependent volume of the vessel. The second integral is taken over the nonlinear waterplane area and will be seen to be the nonlinear extension of the Froude-Krylov hydrostatic-like restoring force acting on a floating vessel. For a submerged body this term vanishes. For a surface piercing body and in the limit of small amplitude waves which are long relative to the dimension of the vessel this term is proportional to the heave hydrostatic restoring coefficient $C_{33}\rho g A_W$, where A_W is the static waterplane area, times the ambient wave amplitude.

By applying Gauss's theorem, the first term may be reduced to a volume integral:

$$\boldsymbol{F}_{\text{F-K}} = \rho \frac{d}{dt} \iiint_{\nabla(t)} dV\, \nabla\phi_I + \rho \frac{d}{dt} \iint_{S_W^I(t)} dS\, \phi_I \boldsymbol{n}'. \quad (15)$$

The first term in (15) is the time rate of change of the linear momentum of all the fluid particles of an ambient wave enclosed by the time dependent volume of the vessel. In long waves the volume integral in (15) may to leading order be approximated by evaluating the ambient wave velocity vector at the centroid of the time dependent volume of the vessel. It is noted that the location of this centroid is time dependent.

The second integral in (15) has a familiar interpretation within linear theory. Recall that the linear dynamic free surface condition takes the form

$$\zeta_I = -\frac{1}{g}\left(\frac{\partial\phi_I}{\partial t}\right)_{Z=0}.$$

Substituting in (15), exchanging the time differentiations with the surface and volume integrations and taking into account that the unit vector points inside the volume we obtain the linearized version of the momentum Froude-Krylov force

$$\begin{aligned} \boldsymbol{F}_{\text{F-K, LINEAR}} = \rho \iiint_{\nabla} dV\, \frac{\partial}{\partial t}\nabla\phi_I \\ + \rho g \boldsymbol{k} \iint_{S_W^I(t)} dS\, \zeta_I. \end{aligned} \quad (16)$$

The first term in (16) is the inertia component of the momentum Froude-Krylov force which is equal to the integral of the acceleration of the ambient wave fluid particles within the linearized volume of the vessel below the calm water surface, multiplied by their density. The second term is the hydrostatic contribution which is proportional to the integral of the ambient wave elevation over the static waterplane area of the vessel. For long waves this integral may be approximated to leading order by the product of the waterplane area and the ambient wave elevation at the origin of the coordinate system. In this limiting case the hydrostatic component of the momentum Froude-Krylov force, per unit ambient wave elevation, reduces to the heave restoring coefficient which appears in the left hand side of the linearized vessel equations of motion. As expected, for submerged bodies the hydrostatic component of the momentum Froude-Krylov force vanishes.

In large amplitude waves the hydrostatic component of the Froude-Krylov force (15) may be comparable to the time dependent buoyancy force (11). Moreover, while the buoyancy force always points vertically upwards, the hydrostatic component of the Froude-Krylov force component has an oblique orientation which is a function of the inclination of the ambient wave surface contained in the unit normal vector. In the limit of linear theory this Froude-Krylov hydrostatic force points vertically upwards.

CONCLUSIONS

A new nonlinear momentum formulation has been developed. This formulation leads to the explicit decomposition of the total hydrodynamic force in nonlinear hydrostatics, Froude-Krylov and wave disturbance forces in steep random waves which are easily amenable to computation. All force components appear as time derivatives of the respective hydrodynamic impulses, defined as spatial integrals of the respective velocity potentials over the vessel instantaneous wetted surface, which do not requite the numerical evaluation of time derivatives of the velocity potential over the vessel wetted surface.

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