Probabilistic Response of Mathieu Equation Excited by Correlated Parametric Excitation

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Abstract

We derive analytical approximations for the probability distribution function (pdf) for the response of the Mathieu equation with random parametric excitation at the main resonant frequency. The inclusion of stochastic excitation renders the otherwise straightforward response into a system exhibiting intermittent resonance. Due to the random amplitude term the system may momentarily cross into the instability region, triggering an intermittent system resonance. As a result, the statistics of the response are characterized by heavy-tails. We develop a mathematical approach to study this problem by conditioning the density of the system response on the occurrence of an instability, and analyze separately the stable and the unstable regimes.

Keywords: *Mathieu equation, random excitation, heavy-tails quantification, intermittent instabilities.*

1 INTRODUCTION

In this work we consider a Mathieu type stochastic differential equation of the form

$$\ddot{x}(t) + 2\varepsilon\zeta_0\omega_0\dot{x}(t) + \omega_0^2(1 + \varepsilon\beta(t)\sin\Omega t)x(t) = \sqrt{\varepsilon}\xi(t), \qquad (1)$$

where ζ_0 is the damping ratio, ω_0 is the undamped natural frequency of the system, Ω is the frequency of the harmonic excitation term and $\beta(t)$ its (random) amplitude, ε is a small positive parameter, and $\xi(t)$ is a broadband weakly stationary random excitation. It is well known that the Mathieu equation

$$\ddot{x}(t) + 2\zeta_0\omega_0\dot{x}(t) + \omega_0^2(1 + \lambda\sin\Omega t)x = 0, \qquad (2)$$

displays instability due to resonance depending upon the parametric excitation frequency and amplitude parameters in the (Ω, λ) plane. Near $\Omega/2\omega_0 = 1/n$ for positive integers n, we have regions of instabilities, with the widest instability region being for $n = 1$. Damping has the effect of raising the instability regions from the $\Omega/2\omega_0$ axis by $2(2\zeta)^{1/n}$. Therefore, for $\zeta \ll 1$ the instability region near $n = 1$ is of greatest practical importance (Lin & Cai, 1995, Nayfeh & Mook, 1984). In the following we consider (1) tuned to the important resonant frequency $\Omega = 2\omega_0$. The case, where the frequency is slightly detuned can be generalized following exactly the same approach, but for simplicity of the presentation we consider no detuning. In realistic systems the parameter λ in (2) that controls the stability of the system for a fixed Ω and ζ_0 may be a random quantity and not necessarily deterministic. If this is the case, intermittent resonant instabilities may occur due to the randomly varying parameter $\beta(t)$ in (1) crossing momentarily into the instability region which induces a short-lived large amplitude spike in the response after which the system is relaxed back to its stable response (Fig. 1). In other words, we are interested in the case where $\beta(t)$ is on average stable, but can momentarily transition into the instability region due to randomness. From an applications standpoint ignoring randomness in

$\beta(t)$, woud severely underestimate the probability for extreme events since the corresponding averaged equation would lead to Gaussian statistics, whereas the original system features heavy-tailed statistics. It is the purpose of this work to quantify the probabilistic response of (1), in other words the probability distribution function (pdf), for the case when $\beta(t)$ is a random quantity. The strategy we employ utilizes a decomposition of the probabilistic system response into stable and unstable regimes, which are then individually analyzed and combined to construct the full distribution of the response.

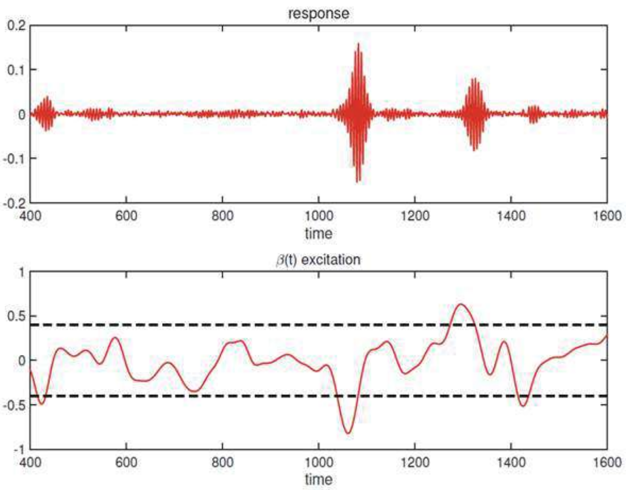

Figure 1 Sample realization of the Mathieu equation (3) (top). The parametric amplitude stochastic excitation term β(t) (bottom) triggers intermittent resonance when it crossing above or below the instability threshold (dashed lines).

2 PROBLEM STATEMENT

We consider the following equation

$$\ddot{x}(t) + 2\zeta_0\omega_0\dot{x}(t) + \omega_0^2(1 + \beta(t)\sin 2\omega_0 t)x(t) = \xi(t), \tag{3}$$

where it is understood that the order of the terms are as in (1), $\xi(t)$ is a broadband weakly stationary excitation, and $\beta(t)$ is a correlated weakly stationary Gaussian process. To derive the probability distribution of (3) the standard approach is by a coordinate transformation to a pair of slowly varying variables and then to apply the stochastic averaging procedure to arrive at a set of Ito stochastic differential equations for the transformed coordinates. The Fokker-Plank equation can then be used to solve for the response pdf (Lin & Cai, 1995, Floris 2012). This standard approach, applied to the problem (3) leads to Gaussian statistics. In reality, randomness in the amplitude $\beta(t)$ leads to intermittent parametric instabilities, and therefore non-Gaussian statistics. To account for the statistics due to intermittent events triggered by $\beta(t)$ we decompose the probabilistic system response into the stable regime and unstable regime according to

$$\mathbf{P}(X) = \mathbf{P}(X \mid \text{stable regime})\mathbf{P}(\text{stable regime}) + \mathbf{P}(X \mid \text{unstable regime})\mathbf{P}(\text{unstable regime}), \tag{4}$$

and derive the corresponding distributions for each term in (4). We assume that instabilities are statistically independent so that the decomposition is applicable; in other words that the frequency of $\beta(t)$ crossing into the instability region is sufficiently rare. We remark that for the system to feature intermittent instabilities it is required that the correlation length of the process $\beta(t)$ must be sufficiently large compared the time scale of damping $\approx 1/\zeta_0$ so that instabilities develop. In the following sections our attention will be aimed for the case where the excitation $\beta(t)$ is described by a *Gaussian* process to facilitate the analytic determination of the terms in (4). However, the ideas developed can be generalized to the case when $\beta(t)$ is a non-Gaussian process by carrying out the procedure using numerically generated realization of the excitation process.

To proceed we will average the governing system (3) over the fast frequency ω_0. We assume that correlation length of the stochastic process $\beta(t)$ varies slowly over the systems natural period $2\pi/\omega_0$ so that $\beta(t)$ can be treated constant over the period, which will be the case in order for (3) to exhibit intermittent instabilities. To apply the method of averaging to (3) we introduce the following transformation

$$x(t) = x_1(t)\cos(\omega_0 t) + x_2(t)\sin(\omega_0 t)$$
$$\dot{x}(t) = -\omega_0 x_1(t)\sin(\omega_0 t) + \omega_0 x_2(t)\cos(\omega_0 t) \quad (5)$$

for the slowly varying variables $x_1(t)$ and $x_2(t)$. Inserting (5) into (3) and using the additional relation $\dot{x}_1\cos(\omega_0 t) + \dot{x}_2\sin(\omega_0 t) = 0$ gives the following pair of differential equations

$$x_1(t) = -[2\zeta_0\omega_0(x_1\sin^2(\omega_0 t) - \frac{1}{2}x_2\sin(2\omega_0 t)) - \frac{\omega_0\beta}{2}x_1\sin^2(2\omega t) - \omega_0\beta x_2\sin^2(\omega_0 t)\sin(2\omega t)] - \frac{1}{\omega_0}\sin(\omega_0 t)\xi(t) \quad (6)$$

$$x_2(t) = [2\zeta_0\omega_0(\frac{1}{2}x_1\sin(2\omega_0 t) - x_2\cos^2(\omega_0 t)) - \frac{\omega_0\beta}{2}x_2\sin^2(2\omega t) - \omega_0\beta x_1\cos^2(\omega_0 t)\sin(2\omega t)] + \frac{1}{\omega_0}\cos(\omega_0 t)\xi(t) \quad (7)$$

Averaging the deterministic terms in brackets in (6) and (7) over the fast frequency $\frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} d\,t$ we have

$$\dot{x}_1 = -\left(\zeta_0 - \frac{\beta}{4}\right)\omega_0 x_1 - \frac{1}{\omega_0}\sin(\omega_0 t)\,\xi(t) \quad (8)$$

$$\dot{x}_2 = -\left(\zeta_0 + \frac{\beta}{4}\right)\omega_0 x_2 + \frac{1}{\omega_0}\cos(\omega_0 t)\,\xi(t). \quad (9)$$

The averaged variables x_1 and x_2 provide an excellent statistical and pathwise approximation to the original system. Applying the stochastic averaging procedure to the random forcing gives the following set of Ito stochastic differential equations for the transformed coordinates

$$\dot{x}_1 = -\left(\zeta_0 - \frac{\beta(t)}{4}\right)\omega_0 x_1 + \sqrt{2\pi K}\,\dot{W}_1(t) \quad (10)$$

$$\dot{x}_2 = -\left(\zeta_0 + \frac{\beta(t)}{4}\right)\omega_0 x_2 + \sqrt{2\pi K}\,\dot{W}_2(t), \quad (11)$$

with $K = S_{\xi\xi}(\omega_0)/2\omega_0^2$, where $S_{\xi\xi}(\omega_0)$ is the spectral density of the additive excitation $\xi(t)$ at frequency ω_0, and $\dot{W}_1$ and $\dot{W}_2$ are independent white noise of unit intensity (Lin & Cai, 1995). The slowly varying coordinates after averaging transform into two decoupled Ornstein-Uhlenbeck (OU) processes. While averaging the forcing term provides poor pathwise agreement with the original system, it does however provide excellent statistical agreement.

3 PROBABILITY DISTRIBUTION OF THE SLOW VARIABLES

Here we will present the main results on how the heavy-tailed statistics of the averaged slowly varying variables in (10) can be approximated by separating the response into stable and unstable regimes according to (4). If $\beta(t)$ is a zero mean process both x_1 and x_2 will follow the same distribution. Incorporating bias in the amplitude excitation term $\beta(t)$ is straightforward. We will however concentrate on a zero mean process which we write as $\beta(t) = k\tilde{\gamma}(t)$, where $\tilde{\gamma}(t)$ is a Gaussian process with zero mean and unit variance. We consider the following OU process which represents x_1 or x_2

$$\dot{x} = -\left(\zeta_0 - \frac{\beta(t)}{4}\right)\omega_0\, x + \sigma\,\dot{W}(t) \quad (12)$$

We write (12) as

$$\dot{x} = -\gamma(t)\,x + \sigma\,\dot{W}(t), \quad (13)$$

so that $\gamma(t) = \tilde{m} - \tilde{k}\tilde{\gamma}(t)$ is a Gaussian process with mean $\tilde{m} = \zeta_0\omega_0$ and standard deviation $\tilde{k} = k\omega_0/4$. From (13) it is clear that intermittency is triggered when γ has zero downcrossings.

We define the threshold of an instability by $\eta = \tilde{m}/\tilde{k}$. So that the probability of γ being in a regime that does not trigger instabilities is given by $\mathbf{P}(\gamma > 0) = \Phi(\eta)$ and otherwise by $\mathbf{P}(\gamma < 0) = 1 - \Phi(\eta)$ (where $\phi(\cdot)$ denotes the standard normal pdf and $\Phi(\cdot)$ denotes the standard normal cdf). However due to the relaxation phase after an instability the probability $\mathbf{P}(\text{unstable regime})$ is not exactly $1 - \Phi(\eta)$. To determine the typical duration of the decay phase, we note that during the growth phase the dynamics are approximately given by $x_p = x_0 e^{-\overline{\gamma}|_{\gamma<0} T_{\gamma<0}}$, where $T_{\gamma<0}$ is the duration for which $\gamma < 0$, x_p is the peak value of x during the instability, and $\overline{\gamma}|_{\gamma<0}$ is the conditional mean of γ given $\gamma < 0$. Similarly, for the decay phase $x_0 = x_p e^{-\overline{\gamma}|_{\gamma>0} T_{\text{decay}}}$. Combining these two results we have that the typical ratio between the growth and decay phase is given by

$$\frac{T_{\text{decay}}}{T_{\gamma<0}} = -\frac{\overline{\gamma}|_{\gamma<0}}{\overline{\gamma}|_{\gamma>0}} = -\frac{\tilde{m} - \tilde{k}\frac{\phi(\eta)}{1-\Phi(\eta)}}{\tilde{m} + \tilde{k}\frac{\phi(\eta)}{\Phi(\eta)}} = \nu. \quad (14)$$

The total duration of an unstable event is thus given by the sum of the duration of $T_{\gamma<0}$ and T_{decay}: $T_{\text{inst}} = (1+\nu)T_{\alpha<0}$. Making this modification we have

$$\mathbf{P}(\text{unstable regime}) = (1+\nu)\mathbf{P}(\gamma < 0), \quad (15)$$

$$\mathbf{P}(\text{stable regime}) = 1 - (1+\nu)\mathbf{P}(\gamma < 0). \quad (16)$$

3.1 Stable Regime Distribution

In the stable regime we have by definition no intermittent events. The dynamics can therefore be well approximated by replacing $\gamma(t)$ by its conditionally stable average $\overline{\gamma}|_{\gamma>0} = \tilde{m} + \tilde{k}\frac{\phi(\eta)}{\Phi(\eta)}$ so that

$$\dot{x} = -\overline{\gamma}|_{\gamma>0}\, x + \sigma \dot{W}(t). \quad (17)$$

The corresponding stationary pdf of (17) is a Gaussian distribution by the Fokker-Planck equation (Soong & Grigoriu, 1993). This gives us the following distribution for the conditionally stable dynamics

$$\mathbf{P}(X = x \mid \text{stable regime}) = \sqrt{\frac{\overline{\gamma}|_{\gamma>0}}{\pi\sigma^2}}\, e^{-\frac{\overline{\gamma}|_{\gamma>0} x^2}{\sigma^2}}. \quad (18)$$

3.2 Unstable Regime Distribution

Next we will derive the pdf for the system response for the unstable regime. In the unstable regime there is a growth phase due to the stochastic process $\gamma(t)$ crossing below the zero level, which triggers the instability. During this stage we assume that the parametric excitation is the primary mechanism driving the instability and ignore the small white noise forcing term which has a negligible minimal impact on the pdf of the response. We characterize the growth phase by the envelope of the response $u \simeq u_0 e^{\Lambda T_{\gamma<0}}$, where u_0 is a random variable that characterizes the stable envelope pdf, Λ is a random variable that represents the Lyapunov exponent, and $T_{\gamma<0}$ is the random length of time that the stochastic process γ spends below the zero level.

We first determine the energy growth distribution $Q = \Lambda T_{\gamma<0}$. By substituting the representation $u \simeq u_0 e^{\Lambda T_{\gamma<0}}$ into (13) we obtain that the eigenvalue is given by $\Lambda = -\gamma$ so that

$$f_\Lambda(x) = \mathbf{P}(-\gamma \mid \gamma < 0) = \frac{1}{k(1-\Phi(\eta))}\phi\left(\frac{x+\tilde{m}}{\tilde{k}}\right). \quad (19)$$

To determine analytically the distribution of the time that the stochastic process $\tilde{\gamma}$ spends below an arbitrary threshold level η is not in general possible. However an asymptotic expression is available for the case of rare crossings $\eta \to \infty$ (Rice 1958)

$$f_{T_{\gamma<0}}(t) = \frac{\pi t}{2\overline{T}^2} e^{-\frac{\pi t^2}{4\overline{T}^2}}, \quad (20)$$

which in our case provides a very good approximation since we assume that instabilities are rare so that η is relatively large. In (20) $\overline{T}$ represents the average length of an instability which for a Gaussian process is given by the ratio between the probability of $\gamma < 0$ and the average number of upcrossings of level η per unit time $\overline{N}^+(\eta)$ (Rice 1958)

$$\overline{T}_{\gamma<0} = \frac{\mathbf{P}(\tilde{\gamma} > \eta)}{\overline{N}^+(\eta)} = \frac{1-\Phi(\eta)}{\frac{1}{2\pi}\sqrt{-R''_{\tilde{\gamma}}(0)}\exp\left(-\frac{\eta^2}{2}\right)}, \quad (21)$$

where we have used Rice's formula for the expected number of upcrossings (Blake & Lindsey, 1973, Kratz 2006) and where $R_{\tilde{\gamma}}(x)$ represents the correlation of the process $\tilde{\gamma}$. With these results we can determine the distribution of the energy excitation statistics by the product distribution

$$f_Q(z) = \int_0^\infty f_\Lambda(x) f_{T_{Y<0}}(z/x) \frac{1}{|x|}\, dx. \quad (22)$$

With the distribution of the energy excitation statistics from (22) we can now derive the pdf for the unstable response. For simplicity let $U = Ye^Q$ and $\Lambda = Q$, so that by a random variable transformation we have

$$\begin{aligned} f_{U\Lambda}(u,\lambda) &= f_{YQ}(y,q)\det|\partial(y,q)/\partial(u,\lambda)| \\ &= \frac{1}{e^Q} f_Y(y) f_Q(q) \\ &= \frac{1}{e^\lambda} f_Y(u/e^\lambda) f_Q(\lambda). \end{aligned} \quad (23)$$

Therefore the general form of the system pdf is given by

$$f_U(u) = \int_0^\infty \frac{1}{e^\lambda} f_Y(u/e^\lambda) f_Q(\lambda)\, d\lambda. \quad (24)$$

Where the pdf f_Y corresponds to the pdf of the initial point of the instability. We note that the OU process has the property that its interaction with the parametric excitation gives rise to "instabilities" of very small intensity which are indistinguishable from the typical stable state response. To enforce the separation of the unstable response from the stable state requires we introduce the following correction to the initial point of the instability $Y = |x_s| + c$, where $|\tilde{x}_s|$ is the pdf of the envelope of the stable response (Rayleigh) and c is a constant that enforces the separation. We find that choosing c such that it is one standard deviation of the typical stable response is sufficient to enforce this separation and works well in practice. In addition, this choice is associated with very robust performance over different parametric regimes. Therefore we have that the distribution of the initial point of an instability is given by

$$f_Y(x) = \frac{2\overline{\gamma}|_{\gamma>0}}{\sigma^2}(x-c)\exp\left(-\frac{\overline{\gamma}|_{\gamma>0}}{\sigma^2}(x-c)^2\right), \quad (25)$$

for $x \geq c$.

Thus after transforming the envelope representation into the full response distribution by a narrowbanded argument we finally have the conditionally unstable distribution

$$\mathbf{P}(X = x \mid \text{unstable regime}) = \frac{1}{2} f_U(|x|) \quad (26)$$

4 COMPARISON WITH MONTE-CARLO EXPERIMENTS

With the results from Section 3 constructing the full probability distribution for the slow variable is straightforward and requires using the result from the unstable regime (26) and stable regime (18) and combining them with the appropriate weights (15) according to the decomposition (4). Since we considered that the noise term $\beta(t)$ is unbiased with zero mean the corresponding distribution for the response of the Mathieu equation (3) will be given by the distribution of the average of the slow variable x_1 and x_2 (which are equivalent) since the response is a narrowbanded process according to $x(t) = x_1(t)\cos(\omega_0 t) + x_2(t)\sin(\omega_0 t)$. This can be seen by considering the probability distribution for $\cos(\varphi)$, where φ is a uniform random variable between 0 and 2π. The pdf for $z = \cos(\varphi)$ is given by $f_z(x) = 1/(\pi\sqrt{(1-x^2)})$, $x \in [-1,1]$, which we approximate by $f_z(x) = \frac{1}{2}(\delta(z+1) + \delta(z-1))$.

For the Monte-Carlo experiments we solve (3) with white noise forcing and non-dimensionalize time by ω_0 so that

$$\ddot{x}(t) + 2\zeta_0\, \dot{x}(t) + (1 + \beta(t)\sin 2t)x(t) = \delta\dot{W}(t), \quad (27)$$

for 2500 realization using forward-Euler integration with a time step $dt = 5 \times 10^{-3}$ from $t = 0$ to $t = 3500$ and discard the first 500 time units to ensure statistical steady states from any initial transients. Moreover we simulate the stochastic process $\beta(t)$ according to the method presented in Percival 1992.

For comparisons we present three cases. Moreover, even in very turbulent regimes with frequent instabilities our results capture the *trend* of the tails. We set the damping at $\zeta_0 = 0.1$, $\delta = 0.002$, the correlation length of $\beta(t)$ to be 50 times the time scale of damping $L_{\text{corr}} = 500$, and show three cases with varying frequency of instabilities by changing the variance of β. For the most intermittent regime we set the standard deviation of $\beta(t) = k\tilde{\gamma}(t)$ to $k = 0.229$ so that rare event crossings occur with a 4.0% chance, for the middle regime $k = 0.200$ with a 2.3% chance of rare event crossings, and finally for the least intermittent regime $k = 0.178$ with a 1.2% rare event crossing frequency, see Fig. 2. Overall we have good agreement between the analytic distribution and Monte-Carlo results for these three cases, we stress that the results are robust across a range of parameters that satisfy the assumptions.

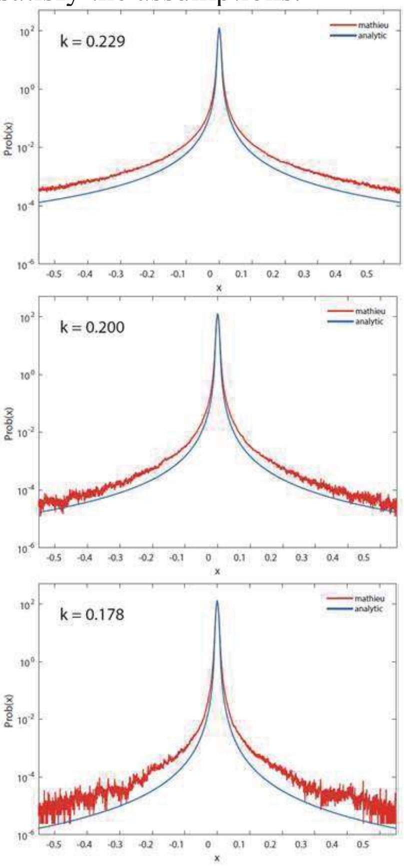

Figure 2 Comparison of Monte-Carlo results of the Mathieu equation (27) (red curve) and analytic probability distribution (blue curve) for various intermittency levels with (left) being most intermittent and (right) least intermittent (semilogarithmic y-axis scale).

5 CONCLUSIONS

In this work we derive an analytic approximation to the pdf for the damped Mathieu equation tuned to the main resonant frequency with random amplitude on the harmonic parametric excitation term. This system features intermittent resonance due the random nature of the amplitude term that triggers intermittent resonance and these intermittent events lead to complex heavy-tailed statistics. To derive the pdf for the response we average the governing equation over the fast frequency to arive at a set of parametrically excited OU processes. We then decompose the response for the slow variables by conditioning on the stable regime and the unstable (transient) state. In the stable regime we employ classical results to describe the pdf of the statistical steady state. In the unstable regime we capture the response by characterizing the transients bursts by an exponential representation with a random Lyapnuov exponent and growth duration. This method allows us to capture the statistics associated with the dynamics that give rise to the heavy-tailed distributions and the resulting analytical approximations compare favorably with direct numerical simulations for a large parameter range.

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6. REFERENCES

Blake, Ian F. and William C. Lindsey (1973). "Level-crossing problems for random processes". In: IEEE Trans. Information Theory IT-19, pp. 295–315.

Floris, Claudio (2012). “Stochastic Stability of Damped Mathieu Oscillator Parametrically Excited by a Gaussian Noise.” In: Mathematical Problems in Engineering 2012, pp. 1–18.

Kratz, Marie F. (2006). “Level crossings and other level functionals of stationary Gaussian processes”. In: Probab. Surv. 3, pp. 230–288.

Lin, Y. K. and C. Q. Cai (1995). Probabilistic Structural Dynamics. McGraw-Hill, Inc.

Nayfeh, A. H. and D. T. Mook (1984). Nonlinear Oscillations. New York: Wiley-Interscience.

Percival, D. B. (1992). “Simulating Gaussian Random Processes with Specified Spectra”. In: Computing Science and Statistics 24, pp. 534–538.

Rice, S. O. (1958). “Distribution of the duration of fades in radio transmission: Gaussian noise model”. In: Bell System Tech. J. 37, pp. 581–635.

Soong, T. and M. Grigoriu (1993). Random Vibration of Mechanical and Structural Systems. PTR Prentice Hall.