Split-time Method for Estimation of Probability of Capsizing Caused by Pure Loss of Stability

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ABSTRACT

The paper reviews a multi-year research effort for using the split-time method to calculate the probability of ship capsizing due to pure loss of stability in irregular waves. The idea of the split-time method is to separate the complex problem of the probability of capsizing into two less complex problems: a non-rare problem that involves the upcrossing of an intermediate level of roll and a rare problem that focuses on capsizing after an upcrossing. An initial implementation using a dynamic model with piecewise linear stiffness, which can be considered to be the simplest model of capsizing in beam seas, led to the concept of critical roll rate as the smallest roll rate at the instant of upcrossing that inevitably leads to capsizing. The piecewise linear system allows a closed-form solution for the critical roll rate, but a more general approach using perturbations can be used for numerical models including advanced hydrodynamic simulation codes. The extension of the split-time method to pure loss of stability required the consideration of the change of roll stiffness in waves and led to calculating the critical roll rate at each upcrossing. A metric of the likelihood of capsizing has been defined as the difference between the observed and critical roll rate at the instances of upcrossing. The probability of capsizing after upcrossing becomes an extrapolation problem for the value of the metric, which can be performed by approximating the tail of the metric's distribution with the Generalized Pareto Distribution. This probability is combined with the observed rate of upcrossings to estimate the rate of capsizing in irregular seas.

Keywords: *Capsizing, Probability, Pure Loss of Stability, Split-Time Method*

1. INTRODUCTION

The dynamic capsizing of a ship is a complex phenomenon dominated by the nonlinearity of the large amplitude roll response, so that linearized mathematical models cannot retain the phenomenon's essential properties. Capsizing of an intact ship in realistic irregular waves represents an even bigger challenge, as this extreme nonlinearity is combined with extreme rarity, leaving no chance for using brute-force Monte-Carlo simulation with hydrodynamic codes of sufficient fidelity.

This challenge has been taken up by the US Office of Naval Research (ONR) project "A Probabilistic Procedure for Evaluating the Dynamic Stability and Capsizing of Naval Vessels". The objective of the project is to create a robust theory of probabilistic capsizing in irregular waves and a numerical procedure based on this theory.

As is well known, an intact ship in realistic ocean waves can capsize in a number of different scenarios or modes. The physical mechanism is different for each scenario, so the theory must be mode-specific. The pure loss of stability is, in a sense, a simplest scenario. It can be modeled in a basin just with waves, assuming that roll damping is high enough to prevent parametric roll resonance and the forward speed is too low for surf-riding and broaching-to.

The split-time method has been developed to simultaneously address the phenomenon's extreme nonlinearity and rarity by providing an evaluation of the probability of capsizing from a relatively small volume of irregular sea response data, perhaps hundreds of hours of simulation rather than the millions of hours required for a Monte-Carlo approach. Numerical implementation has largely been carried out using the Large Amplitude Motion Program (LAMP), although the procedure is fundamentally code-independent.

The split-time method is being developed in phases. The initial phase considered a ship with time-invariant stiffness. While this can be considered to be a model of capsizing at zero speed in beam seas, the primary objective was to develop a basic theory of the method for both a simplified mathematical model and numerical simulation codes. The second phase of the development has extended the theories to the problem of pure loss of stability by considering the ship's change of stiffness as it moves in waves. This paper discusses the development of the theory for these initial two phases.

2. BASIC THEORY OF RARE RANDOM TRANSITIONS

2.1 Piecewise Linear System

Capsizing can be considered as a transition of a ship moving about its stable upright equilibrium to motions about its stable "mast down" equilibrium. A dynamical system with a piecewise linear stiffness is, possibly, the simplest way to describe a transition between two stable equilibria:

$$\ddot{\phi} + \omega_\phi^2 f_L^*(\phi) = 0 \qquad (1)$$

ω_ϕ is a natural frequency of roll and f^*_L is a piecewise linear stiffness function. As illustrated in Figure 1, equation (1) models the phase plane topology of a ship in calm water, and has a closed-form solution for each range.

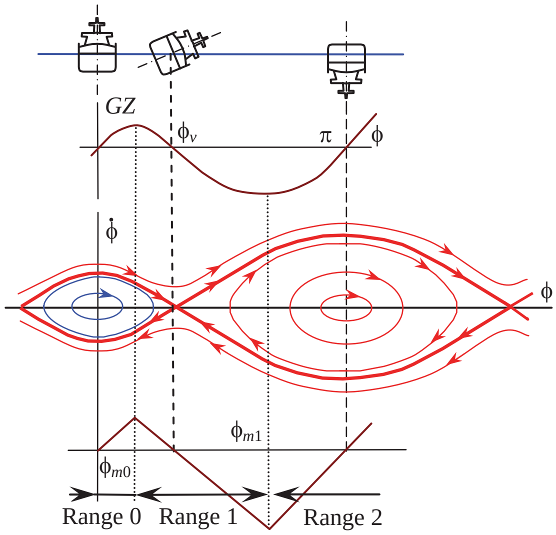

Figure 1: Phase plane topology of capsize and piecewise linear stiffness (Belenky, 1993)

Adding linear damping and random excitation to the dynamical system (1) makes it a model of random transition between stable equilibria:

$$\ddot{\phi} + 2\delta\dot{\phi} + \omega_\phi^2 f_L^*(\phi) = f_{E\phi}(t) \qquad (2)$$

δ is a linear damping coefficient and $f_{E\phi}$ is a stochastic process of roll excitation, modeled as:

$$f_{E\phi}(t) = \sum_{i=1}^{N_\omega} \alpha_{Ei} \sin(\omega_i t + \varphi_{0i}) \qquad (3)$$

α_{Ei} are amplitudes, ω_i are frequencies and φ_{0i} are initial phases of the i^{th} component of an excitation process presented as a Fourier series with N_ω frequencies. Equation (2) has a known closed-form solution in each range:

$$\phi = \begin{cases} \phi_a e^{-\delta t}\sin(\omega_{d0}t+\varepsilon)+p_0(t) \\ \qquad if \quad -\phi_{m0} \le \phi \le \phi_{m0} \\ Ae^{\lambda_1 t}+Be^{\lambda_2 t}+p_1(t) \\ \qquad if \quad \phi_{m0} < \phi \le \phi_{m1} \\ \phi_{a2}e^{-\delta t}\sin(\omega_{d2}t+\varepsilon_2)+p_2(t) \\ \qquad if \quad \phi > \phi_{m0} \end{cases} \tag{4}$$

ϕ_a, ε, A, B, ϕ_{a2} and ε_2 are arbitrary constants that are dependent on the initial conditions at the "switching" of the ranges; ω_{d0} and ω_{d2} are frequencies of the damped oscillation in ranges 0 and 2, respectively; λ_1 and λ_2 are eigenvalues for the solution in Range 1. The particular solutions for each range are expressed as:

$$p_j(t) = \sum_{i=1}^{N_\omega} p_{ij}\sin(\omega_i t + \beta_{ij} + \varphi_{0i}) + E_j \tag{5}$$

j=0, 1, 2 is a range number, p_{ij} is an amplitude and β_{ij} is a phase shift of the i^{th} component of the response. E_j is a position of equilibria for each range:

$$E_0 = 0\,; \quad E_1 = \phi_V\,; \quad E_2 = \pi \tag{6}$$

ϕ_V is the angle of vanishing stability. One of the eigenvalues for the Range 1 is positive while another is negative:

$$\begin{aligned} \lambda_1 &= -\delta + \sqrt{\omega_\phi^2 k_{f1} + \delta^2} > 0 \\ \lambda_2 &= -\delta - \sqrt{\omega_\phi^2 k_{f1} + \delta^2} < 0 \end{aligned} \tag{7}$$

k_{f1} is the slope coefficient for Range 1 taken with opposite sign.

2.2 Condition of the Transition

Whether the transition to the "mast down" equilibrium (*i.e.* capsize) occurs is determined by the sign of the arbitrary constant A, as the first term in Range 1 in solution (4) is unlimited (for non-zero A):

$$A = \frac{(\dot\phi_1 - \dot p_{01}) - \lambda_2(\phi_{m0} - p_{01} - \phi_V)}{\lambda_1 - \lambda_2} \tag{8}$$

$\dot\phi_1$ and ϕ_{m0} are initial conditions, and $\dot p_{01}$ and p_{01} are values of particular solution (5) at the instant of crossing from Range 0 into Range 1. If A>0, the transition occurs immediately, as illustrated in Figure 2. One can express the condition of transition in terms of the roll rate at the instant of upcrossing ϕ_{m0}:

$$\dot\phi_{cr} = \lambda_2(\phi_{m0} - p_{01} - \phi_V) + \dot p_{01} \approx -\lambda_2\phi_V \tag{9}$$

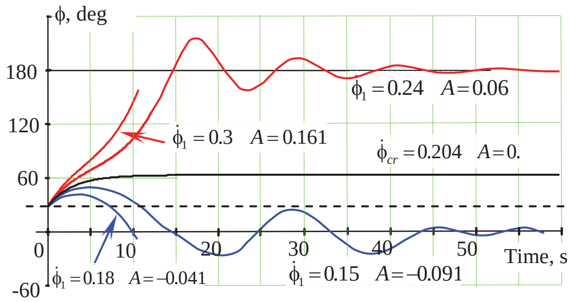

Figure 2: General solution of homogenous equation (2), the derivative values are expressed in rad/s

Values of a particular solution and its derivative at the instant of upcrossing are small and can be neglected. The dynamical system (4) is a repeller in Range 1, so resonance is impossible and the particular solution is small; see Figure 3. The same argument can be applied to the value of the derivative of the particular solution in Range 1 (Belenky, 1993).

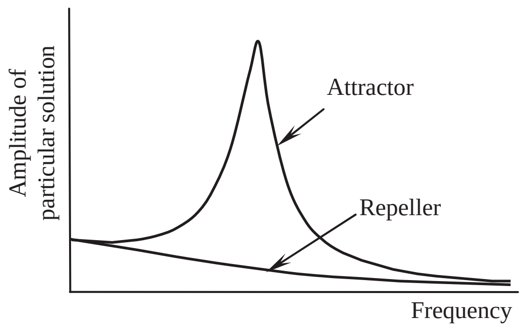

Figure 3: Amplitudes of frequency-domain response of attractor and repeller (Belenky *et al.*, 2008)

Thus, the random transition occurs whenever the process upcrosses the threshold ϕ_{m0} and its derivative exceeds the critical value (9) at the instant of upcrossing. Some upcrossings will result in the transition, while other will not. The transition can be seen as an upcrossing, with its rate reduced by the probability of the derivative's exceeding the critical value (9):

$$\xi_C = \xi_U(\phi_{m0}) \cdot P(\dot{\phi}_1 > \dot{\phi}_{Cr}) \qquad (10)$$

where $\xi_U(\phi_{m0})$ is the upcrossing rate through the threshold ϕ_{m0}.

Formula (10) expresses the main idea of the split-time method. The complex problem of transition has been divided into two less complex problems: characterizing the upcrossing of the intermediate level (non-rare problem) and finding the probability of transition if the upcrossing has occurred (rare problem).

2.3 The Non-Rare Problem

The random transitions (capsizings) are expected to be rare. If one assumes that upcrossings of the threshold ϕ_{m0} are also rare, then influence from the general solution of the homogenous equation on Range 1 can be neglected. For this model, the process of the particular solution is normal and the rate of upcrossing can be expressed as:

$$\xi_U(\phi_{m0}) = \frac{1}{2\pi}\sqrt{\frac{V_{\dot{\phi}}}{V_{\phi}}}\exp\left(-\frac{\phi_{m0}^2}{2V_{\phi}}\right) \qquad (11)$$

$V_{\phi} = V_{p0}$ and $V_{\dot{\phi}} = V_{\dot{p}0}$ are the variances of the particular solution and its derivative in Range 0, and can be found from formula (5):

$$V_{p0} = \sum_{i=1}^{N_\omega} p_{0i}^2 \quad ; \quad V_{\dot{p}0} = \sum_{i=1}^{N_\omega} \omega_i p_{0i}^2 \qquad (12)$$

The calculation of these variances does not present a problem.

2.4 The Rare Problem

The difference between the observed and critical values of the derivative is the metric of the transition's likelihood. Since the critical value for the derivative is constant, it is only necessary to find the distribution of the value of the derivative at upcrossing. The original derivation was published in Appendix 3 of Belenky *et al.* (2008), while the abridged and updated version is given below.

The upcrossing event is defined as follows (Kramer and Leadbetter, 1967):

$$U = \begin{cases} \phi(t) < \phi_{m0} \\ \phi(t+dt) > \phi_{m0} \end{cases} = \begin{cases} \phi(t) < \phi_{m0} \\ \phi(t) > \phi_{m0} - \dot{\phi}(t)dt \end{cases} \qquad (13)$$

The probability that U occurs at time t is

$$P(U) = \int_0^{\infty} \int_{\phi_{m0}-\dot{\phi}dt}^{\phi_{m0}} pdf(\phi,\dot{\phi})\,d\phi\,d\dot{\phi} = dt\int_0^{\infty} \dot{\phi}\cdot pdf(\phi=\phi_{m0},\dot{\phi})\,d\phi\,d\dot{\phi} \qquad (13)$$

$pdf(\phi,\dot{\phi})$ is the joint probability distribution function of the process and its derivative.

Consider a random event V such that:

$$V = \begin{cases} \dot{\phi}(t) \le \dot{\phi}_1 \\ \dot{\phi}(t) > 0 \end{cases} \qquad (15)$$

A random event that the events U and V occur simultaneously:

$$U \cap V = \begin{cases} \phi(t) < \phi_{m0} \\ \phi(t) > \phi_{m0} - \dot{\phi}(t)dt \\ \dot{\phi}(t) \le \dot{\phi}_1 \\ \dot{\phi}(t) > 0 \end{cases} \qquad (16)$$

The probability that U and V occur simultaneously at time t is:

$$P(U \cap V) = \int_0^{\dot\phi_1} \int_{\phi_{m0}-\dot\phi dt}^{\phi_{m0}} pdf(\phi, \dot\phi) d\phi d\dot\phi = dt \int_0^{\dot\phi_1} \dot\phi \cdot pdf(\phi = \phi_{m0}, \dot\phi) d\dot\phi \quad (17)$$

By definition, the Cumulative Distribution Function (*CDF*) is:

$$CDF\left(\dot\phi_1 \mid \phi_{m0}\right) = P\left(\dot\phi(t) < \dot\phi_1 \mid \phi_{m0}\right) = \frac{P(V \cap U)}{P(U)} = \frac{\int_0^{\dot\phi_1} \dot\phi \cdot pdf(\phi = \phi_{m0}, \dot\phi) d\dot\phi}{\int_0^{\infty} \dot\phi \cdot pdf(\phi = \phi_{m0}, \dot\phi) d\dot\phi} \quad (18)$$

Differentiation of (18) yields the *pdf* of the derivative value at the instant of upcrossing:

$$pdf(\dot\phi_1 \mid \phi_{m0}) = \frac{\dot\phi_1 \cdot pdf(\phi = \phi_{m0}, \dot\phi_1)}{\int_0^{\infty} \dot\phi \cdot pdf(\phi = \phi_{m0}, \dot\phi) d\dot\phi} \quad (19)$$

If upcrossings are rare, the response process and its derivative can be assumed to be normal. This also means that they are independent, as the stationary process is not correlated with its first derivative and two uncorrelated normal processes are independent. Substitution of the normal distribution into (19) yields:

$$pdf(\dot\phi_1) = \frac{\dot\phi_1}{V_{\dot\phi}} \exp\left(-\frac{\dot\phi_1^2}{2V_{\dot\phi}}\right) \quad (20)$$

Formula (20) is the Raleigh distribution.

The distribution of the derivative at the instant of upcrossing is different from the distribution of the derivative in general. The distribution "in general" is obtained if the sampling is done in "any" (or random) instant. The instant of upcrossing of the primitive is not a random instant. A condition when the upcrossing is occurred is expressed by equation (13). Thus, the distribution of the derivative at upcrossing is not equivalent to the distribution of the derivative "in general".

Finally, the conditional probability of the transition after upcrossing has occurred is derived using equations (9) and (20):

$$P\left(\dot\phi_1 > \dot\phi_{Cr}\right) = \exp\left(-\frac{(\lambda_2 \phi_V)^2}{2V_{\dot\phi}}\right) \quad (21)$$

Equation (20) is the solution of the rare problem.

2.5 Probability of Rare Transitions

The combination of equations (10) and (20) yields the solution for the rate of rare random transitions:

$$\xi_C = \frac{1}{2\pi} \sqrt{\frac{V_{\dot\phi}}{V_\phi}} \exp\left(-\frac{1}{2}\left(\frac{\phi_{m0}^2}{V_\phi} + \frac{(\lambda_2 \phi_V)^2}{V_{\dot\phi}}\right)\right) \quad (21)$$

For the ship-like dynamical system with piecewise linear stiffness (Figure 1), the domain of attraction to the capsized equilibrium is larger than for the one with "mast up". Thus, while the transition to capsized equilibrium is rare, the probability of transition in the opposite direction can be neglected. It is safe to assume that once transition has occurred, the dynamical system will stay capsized. That means that any two transitions are independent, as they must occur in two independent records. This means the transition meets the requirement of Poisson flow (Sevastianov, 1963, 1994), which leads to the following formula for the probability of transition (capsizing) during a given time *T*:

$$P(T) = 1 - \exp\left(-\xi_C T\right) \quad (22)$$

2.6 Summary of the Basic Theory

The original solution for random rare transitions in a dynamical system with piecewise linear stiffness was found in the late 1980s (Belenky, 1993). It has been applied to the probability of capsizing of a ship in beam wind and seas (Paroka and Umeda, 2006; Paroka *et al*., 2006). Some verification of self-consistency was carried out within the framework of the ONR project, which resulted in the refinement of the solution; the distribution of the derivative value at the instant of upcrossing was found to be Rayleigh (Belenky *et al*., 2008).

The dynamical system with the piecewise linear stiffness likely represents the simplest model of a rare random transition between two stable equilibria. Nevertheless, considering this simple model, the following conclusions can be reached:

- A "critical derivative" ("critical roll rate") can be defined as the value of the derivative which, if exceeded at upcrossing, inevitably leads to transition (capsizing)
- The difference between the observed and critical derivatives (roll rates) can be used as a metric of the likelihood of transition (capsizing)
- The rate of transitions (capsizings) can be defined as the rate of upcrossings of a maximum stiffness level in which the observed derivative (roll rate) exceeds the critical derivative (roll rate)
- The calculations of upcrossings and critical roll rate can be considered separately as non-rare and rare problems, respectively.

3. NUMERICAL EXTENSION OF THE BASIC THEORY OF RARE RANDOM TRANSITIONS

3.1 Toward a Time-Domain Solution

The dynamical system with piecewise linear restoring (2) yields a closed-form solution for the probability of random rare transition (21), which is the simplest mathematical model of a ship capsizing in waves. Is it possible to apply the split-time method if a dynamical system is represented by a time-domain hydrodynamic simulation code?

The non-rare problem can be readily solved in the time domain, as long as the code can provide a sufficient statistical sample. If the intermediate threshold is set appropriately, one can count the upcrossings and estimate the upcrossing rate and an average number of events per unit of time.

The rare problem can also be recast in the time domain. If one assumes that roll stiffness of the dynamical system does not change in time and can be represented by the calm water GZ curve, then the critical roll rate can be found by an iterative set of numerical simulations. The calculations start at the intermediate level and the initial roll rate is perturbed for each run until capsizing is observed. The iteration scheme will create a picture similar to the one in Figure 2.

The distribution of the roll rate at upcrossing can be estimated statistically. Because capsizing is a rare event, the observed roll rates are expected to be smaller than the critical roll rate. The solution of the rare problem in the numerical case therefore involves statistical extrapolation, so only the tail of the distribution needs to be modeled. Generalized Pareto Distribution can be used for this purpose.

3.2 Numerical Non-Rare Problem

The non-rare problem will be solved using a set of time-domain simulations in "random" realizations of stationary irregular waves, which will typically be derived by discretizing an ocean wave spectrum into a set of component wave frequencies with pseudo-random phases. For the upcrossing rate to be estimated correctly, this model of wave excitation should be statistically representative for the duration of each record. To ensure this, the frequency set must be selected so as to avoid a possible self-repeating effect (Belenky, 2011). Since long records require a very large number of incident wave frequencies to avoid this effect, it is generally more efficient to use a number of relatively short records – about 30 minutes each – than a smaller number of long records.

Following the approach developed for the model with piecewise linear stiffness, the level of the intermediate threshold is set to the maximum of the roll restoring (GZ) curve, where the slope of the curve becomes small enough that the corresponding instantaneous frequency does not support resonance under realistic wave excitation. Wave excitation will then add little energy to the dynamical system after this threshold is exceeded, which justifies the solution of the rare problem for the critical roll rate value without excitation, *i.e*. in clam water.

Once the non-rare simulations are completed, the upcrossing rate is estimated as:

$$\hat{\xi} = \frac{N_U}{N_T \Delta t} \tag{23}$$

N_U is the observed number of upcrossings, N_T is total number of data points in all records, and Δt is the time increment (data sampling rate), which is assumed to be the same for all records.

This estimate is a random number and requires an evaluation of statistical uncertainty. Assuming independence of upcrossings (for the purposes of statistical uncertainty assessment only), the occurrence of an upcrossing at a particular time step can be seen as a Bernoulli trial. The number of observed upcrossings then has a binomial distribution:

$$pmf(N_U) = \binom{N_T}{N_U} p^{N_U} (1-p)^{N_T - N_U} \tag{24}$$

p is a parameter of binomial distribution that has the meaning of the probability of a “success” (*i.e*. upcrossing) at a particular time increment. It can be estimated as:

$$\hat{p} = \frac{N_U}{N_T} \tag{25}$$

The boundary of the confidence interval corresponding to a confidence probability P_β can be computed as:

$$\hat{\xi}(low, up) = \frac{Q_{Bin}\left(0.5(1 \mp P_\beta) \mid \hat{p}\right)}{N_T \Delta t} \tag{26}$$

Q_{Bin} is the quantile of the binomial distribution. Its calculation, however, may encounter numerical difficulties as the total number of points N_T may be large. If this is the case, a normal approximation of the binomial distribution can be used with the following variance estimate:

$$V\hat{a}r(\hat{\xi}) = \frac{\hat{\xi} \cdot (1 - \hat{\xi})}{N_T \Delta t} \tag{27}$$

The boundaries of the confidence interval are then expressed as:

$$\hat{\xi}(low, up) = \hat{\xi} \pm K_\beta \sqrt{V\hat{a}r(\hat{\xi})} \tag{28}$$

K_β is 0.5(1+ P_β) quantile of a standard normal distribution. Further details may be found in Belenky *et al.* (2008) and Campbell and Belenky (2010).

3.3 Numerical Rare Problem

As described above, the numerical solution of the rare problem starts with an iterative set of simulations with different initial conditions to compute the critical roll rate at upcrossing that leads to capsizing. These simulations must include an accurate calculation of the restoring at large roll angles.

The numerical solution of the rare problem was implemented using the Large Amplitude Motion Program (LAMP). LAMP is a mature hybrid time-domain code (Lin and Yu, 1990) incorporating a number of hydrodynamic modeling options of different fidelity.

Figure 4 shows a sample rare solution using a simplest option based on hydrostatics only solution (LAMP-0) with the following features:

- 3-D hydrostatics up to the instantaneous waterline
- Does *not* solve the wave-body disturbance problem
- "Tunable" terms for viscous damping and wave interaction effects (*e.g.* added mass)
- CPU time per 2.5-minute simulation: ~3 seconds.

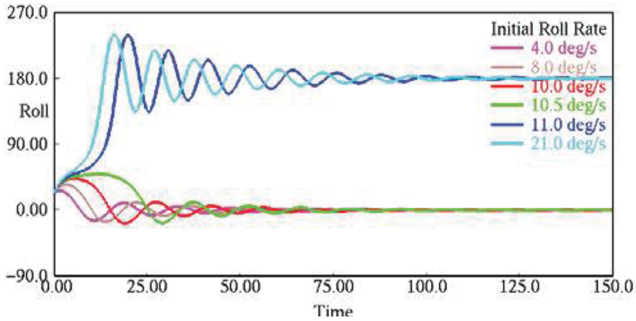

Figure 4: Calculation of critical roll rate via iterative numerical simulation (LAMP-0)

Figure 5 shows the next level of complexity: the approximate body-nonlinear solution (LAMP-2). LAMP-2 is characterized by the following features:

- 3-D hydrostatics up to the instantaneous waterline
- Solves the wave-body disturbance potential over the mean wetted hull surface
- "Tunable" damping terms for viscous effects
- CPU time per 2.5-minute simulation: ~2 minutes (Direct) or ~8 seconds (pre-computed impulse response functions for disturbance potential).

Figure 6 shows the complete body-nonlinear solution (LAMP-4), which is characterized by the following features:

- 3-D hydrostatics up to the instantaneous waterline
- Solves the wave-body disturbance potential over the instantaneous wetted hull surface
- "Tunable" damping terms for viscous effects
- CPU time per 2.5 minute simulation: ~3 hours.

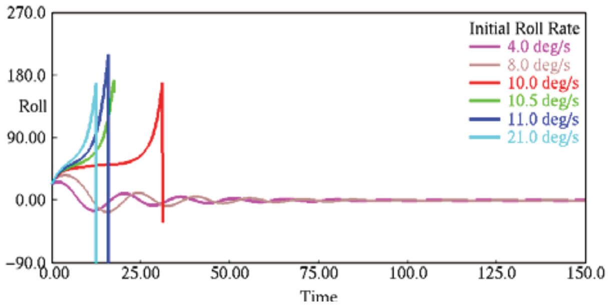

Figure 5: Calculation of critical roll rate via iterative numerical simulation (LAMP-2)

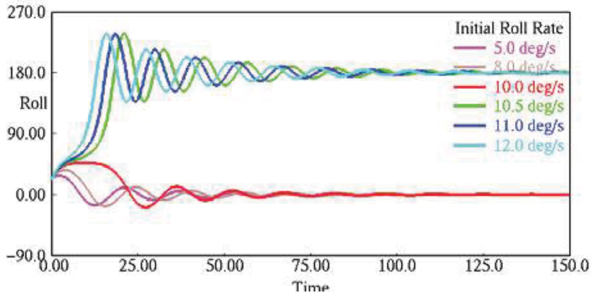

Figure 6: Calculation of critical roll rate via iterative numerical simulation (LAMP-4)

The next step in solving the rare problem is estimating the probability of capsizing after upcrossing. It is associated with exceedance of the critical value by the roll rate at the instant of upcrossing. Its distribution is not known, however, and must be modeled based on the time-domain results. If a good model were available for the joint distribution of roll and roll rate, formula (19) could be used to derive the distribution of the roll rate at the instant of upcrossing; however, no such model is available. Instead, one can attempt to directly model this distribution using the upcrossing data. As the critical roll rate is relatively large, only the tail of the distribution needs to be modeled. Since direct observation of capsizing is not expected, modeling the tail of the distribution is, in fact, a statistical extrapolation problem.

The mathematical background of statistical extrapolation is based on two extreme value theorems (Coles, 2001):

- Fisher-Tippet-Gnedenko theorem states that the largest value of independent identically distributed (IID) random variables asymptotically tends to the Generalized Extreme Value (GEV) distribution
- Pickands-Balkema-de Haan theorem states that the tail of IID random variables can be approximated with the Generalized Pareto Distribution (GPD).

The *pdf* of the GPD is expressed as:

$$pdf(\dot{\phi}_1) = \begin{cases} \dfrac{1}{\sigma}\left(1 + k\dfrac{\dot{\phi}_1 - \mu}{\sigma}\right)^{-\left(1+\frac{1}{k}\right)}; \\ if \quad \mu < \dot{\phi}_1, \quad k > 0, \quad or \\ \qquad \mu < \dot{\phi}_1 < \mu - \dfrac{\sigma}{k}, \quad k < 0 \\ \dfrac{1}{\sigma}\exp\left(-\dfrac{\dot{\phi}_1 - \mu}{\sigma}\right) \quad ; \\ if \quad \mu < \dot{\phi}_1, \; k = 0 \end{cases} \qquad (29)$$

k is the shape parameter, σ is the scale parameter, and μ is the threshold above which GPD is applicable.

These three parameters must be estimated in order to approximate the tail. Belenky *et al.* (2014) describes the technique for fitting the GPD for the more complex case that accounts for stability variation, which will be reviewed later in this paper. One can see that the constant stiffness is the particular case where the critical roll rate remains the same for each upcrossing. There seems to be no reason to believe that the fitting technique will not work for this particular case, as it worked for more general case.

The fitting procedure consists of the following steps (see Belenky *et al.* (2014) for details):

- Set a series of thresholds μ for the observed roll rates at upcrossing – this threshold is the value of the roll rate at which the GPD becomes valid (*i.e.* the start of the tail) and is not to be confused with the intermediate threshold for roll angle
- Use the log-likelihood method (Grimshaw, 1991) to find the estimate of shape and scale parameter for each threshold μ
- Using the Delta method, find variances and covariances of the shape and scale parameter for each threshold μ (Boos and Stefanski, 2013)
- Find the minimum threshold μ providing applicability of GPD, using techniques described in Coles (2001) and based on Reiss and Thomas (2007).

The probability of capsizing if the threshold μ is exceeded is then expressed as:

$$\hat{P}_1 = \begin{cases} \left(1 + \hat{k}\cdot\hat{\sigma}^{-1}\cdot\left(\dot{\phi}_{cr} - \mu\right)\right)^{-\hat{k}^{-1}} \\ \qquad if \;\; \hat{k} > -\dfrac{\hat{\sigma}}{\dot{\phi}_{cr} - \mu} \\ 0 \quad otherwise \end{cases} \qquad (30)$$

The probability (30) is computed using estimates, which are random numbers, so the result of (30) is also an estimate and a random number. It can also be considered as the most probable value, because the scale and shape parameters were estimated with the log likelihood method, *i.e.* they are the most probable values for the parameters.

The next step is evaluating the confidence interval for the probability estimate (30). This is done by considering it as a deterministic function of random arguments. Assuming a normal distribution for the estimates of shape parameter and the logarithm of scale parameter (scale parameter is always positive), the following formula was derived for the distribution of the estimate of probability of capsizing after upcrossing:

$$pdf_{P1}(P_1) = \int_{-\infty}^{\infty} f_N\left(k, \frac{k(\dot{\phi}_{cr}-\mu)}{P_1^{-k}-1}\right) \cdot \frac{k^2 P_1^{-k-1}}{(P_1^{-k}-1)^2}(\dot{\phi}_{cr}-\mu)dk \quad (31)$$

where $f_N(k, \ln(\sigma))$ is a normal joint distribution of the shape parameter and the logarithm of the scale parameter. The boundaries of the confidence interval are computed with the quantiles of the distribution (31). The most probable value of P_1 may be zero; however, this does not necessarily mean that the upper boundary of the confidence interval is zero.

The GPD distribution approximates a tail of the distribution when it exceeds the threshold μ. Equation (31) therefore estimates a conditional probability under the condition that the threshold was exceeded, so the solution of the rare problem is expressed as:

$$\hat{P}(\dot{\phi}_1 > \dot{\phi}_{Cr}) = \frac{N_\mu}{N_U}\hat{P}_1 \quad (32)$$

N_μ is the number of upcrossings when the roll rate has exceeded the threshold μ. The complete estimate of the capsizing rate is:

$$\hat{\xi}_C = \frac{N_U}{N_T \Delta t} \cdot \frac{N_\mu}{N_U}\hat{P}_1 = \frac{N_\mu}{N_T \Delta t} \cdot \hat{P}_1 \quad (33)$$

The number of upcrossings of the intermediate threshold has disappeared from equation (33). Thus, the choice of the intermediate threshold μ can only affect the independence of the upcrossings as a condition of the GPD's applicability.

The fraction in equation (33) is the estimate of the rate of events: upcrossings of the intermediate level when the roll rate has exceeded μ. The confidence interval of this estimate can be computed using formula (26), but N_U must be substituted N_μ:

$$\hat{\xi}_\mu(low, up) = \frac{Q_{Bin}(0.5(1 \mp P_{\beta 1}) \mid \hat{p}_\mu)}{N_T \Delta t}$$
$$\hat{p}_\mu = \frac{N_\mu}{N_T} \quad (34)$$

$P_{\beta 1}$ is the "new" confidence probability; it reflects the fact that the estimate of capsizing rate is a product of two random numbers, each of which has its own confidence interval. As these numbers are independent,

$$P_{\beta 1} = \sqrt{P_\beta} \quad (35)$$

The confidence interval of the estimate P_1 must therefore use $P_{\beta 1}$ as a confidence probability:

$$\hat{P}_1(low, up) = Q_{P1}(0.5(1 \mp P_{\beta 1})) \quad (36)$$

Q_{P1} is the quantile of the distribution (29). Finally, the boundaries of the confidence interval for the capsizing rate estimate (33) are:

$$low(\hat{\xi}_C) = low(\hat{\xi}_\mu) \cdot low(\hat{P}_1)$$
$$high(\hat{\xi}_C) = high(\hat{\xi}_\mu) \cdot high(\hat{P}_1) \quad (37)$$

3.4 Summary for the Numerical Extension of the Basic Theory

The numerical extension of the basic probabilistic theory of capsizing was published in Belenky *et al*. (2008), where most of the numerical problem's specifics were formulated. However, the problem of modeling the distribution of the roll rate remained without practical solution until the applicability of the GPD was fully appreciated (Belenky *et al*., 2014).

The extension demonstrated that the split-time method is applicable for a dynamical system presented by an advanced hydrodynamic simulation code instead of an ordinary differential equation (ODE). The simplest numerical extension involves:

- Assuming that the roll stiffness of the dynamical system can be modeled by the GZ curve in calm water
- Finding the critical roll rate by a series of iterative simulations starting from an intermediate threshold with different rates; the critical roll rate is defined as the largest roll rate not leading to capsizing
- Modeling the tail of the distribution of roll rate at upcrossing with GPD
- Evaluating the statistical uncertainty for the estimates of upcrossing rate and probability of capsizing after upcrossing.

The transition to the numerical solution involves working with time-domain data and requires statistical methods to handle the results of numerical simulation, including the modeling of distributions and the assessment of statistical uncertainty.

4. BASIC THEORY OF RARE RANDOM TRANSITIONS WITH RANDOM STIFFNESS

4.1 Piecewise Linear System

The next step is to find out if the assumption of time-invariant stiffness may be abolished and if a solution can still be obtained in the simplest case with random stiffness. Consider the dynamical system (2), but with the stiffness in Range 1 now time dependent; its intercept is random, but the slope remains the same, as shown in Figure 7.

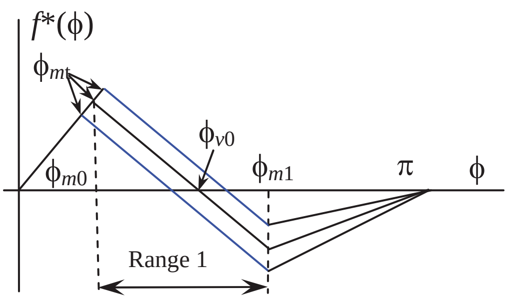

Figure 7: Time-variant piecewise linear stiffness

The variation of the stiffness in waves can be caused by the wave pass effect and ship motions. Both phenomena have certain inertia, so the parameters of time-varying stability are described by stochastic processes rather than random variables. Belenky *et al.* (2011) describes a simple mathematical model where the intercept in Range 1 is a linear function of heave:

$$\ddot{\phi} + 2\delta_{\phi}\dot{\phi} + \omega_{\phi}^2 f^*(\phi, \zeta_G) = f_{E\phi}(t) \quad (38)$$

ζ_G is the heave displacement modeled with a linear ODE. The boundary ϕ_m between Ranges 0 and 1 in equation (38) now depends on time. However, within Range 0 equation (38) is identical to equation (2). The difference caused by the stiffness variation in Range 1 makes equation (38) appear as follows:

$$\begin{aligned} &\ddot{\phi} + 2\delta\dot{\phi} + \omega_{\phi}^2 k_{f1}\phi \\ &\quad = \omega_{\phi}^2 k_b \zeta_G(t) + \omega_{\phi}^2 k_{f1}\phi_{v0} + f_{E\phi}(t) \end{aligned} \quad (39)$$

- Coefficient k_b reflects the dependence of the intercept on the heave displacement ζ_G. The term containing ζ_G is the only difference between (39) and (2).

Taking into account that the slope coefficient k_{f1} has been taken with the opposite sign, equation (39) describes a repeller and its general solution is:

$$\begin{aligned} \phi(t) &= A\exp(\lambda_1 t) + B\exp(\lambda_2 t) \\ &\quad + p_1(t) + \phi_{v0} \end{aligned} \quad (40)$$

The difference between solutions (40) and (4) in Range 1 is the particular solution that now includes the influence of the random variation of stiffness.

4.2 Condition of Transition

The homogenous part of equation (39) does differ from the homogenous part of equation (2) in Range 1, but the condition of transition at the instant of upcrossing t_1 is still the same:

$$A(t_1) > 0 \quad \Rightarrow \quad \dot{\phi}(t_1) > \dot{\phi}_{cr}(t_1) \qquad (41)$$

The critical roll rate is still defined by the same formula (9), but the particular solution can no longer be neglected. As a result, the critical roll rate becomes a function of time, *i.e.* it is a stochastic process:

$$\dot{\phi}_{cr}(t) = \lambda_2 \left(\phi_m(t) - p_1(t) - \phi_{v0} \right) + \dot{p}_1(t) \qquad (42)$$

4.3 Non-Rare Problem

The boundary between Ranges 0 and 1 is now time-dependent, so it makes sense to formulate the non-rare problem for a difference between the boundary and the roll motion, introducing a new stochastic process:

$$x(t) = \phi(t) - \phi_m(t) + \phi_{m0} \qquad (43)$$

Upcrossing of the process $x(t)$ through the threshold ϕ_{m0} makes the switch from Range 0 to Range 1.

The time-dependent boundary $\phi_m(t)$ is a linear function of the heave motion, which is also linear. A Fourier series presentation for $x(t)$ is available from Belenky *et al.* (2011), which allows the upcrossing rate to be expressed using formula (11):

$$\xi_U(\phi_{m0}) = \frac{1}{2\pi} \sqrt{\frac{V_{\dot{x}}}{V_x}} \exp\left(-\frac{\phi_{m0}^2}{2V_x} \right) \qquad (44)$$

V_x and $V_{\dot{x}}$ are variances of the process $x(t)$ and its derivative.

4.4 Rare Problem

To formulate the rare problem, consider a difference between the critical roll rate and the instantaneous roll rate $y(t)$:

$$y(t) = \dot{\phi}_{cr}(t) - \dot{\phi}(t) \qquad (45)$$

The process $y(t)$ is a linear combination of normal processes and can be presented with a Fourier series (Belenky *et al.*, 2011). The capsizing event is associated with a negative value of y at the instant of upcrossing. One therefore needs to find the distribution of the process $y(t)$ at the instant when the dependent process $x(t)$ has an upcrossing. The problem is similar to the one considered in Section 2.4, but instead of a derivative, a dependent process is considered.

To derive the distribution of the process $y(t)$ at the instant of upcrossing, consider a random event W:

$$W = \left(y(t) \le y_C \right) \qquad (46)$$

The events of U (defined by equation (13), but re-formulated for the process $x(t)$) and W occur simultaneously:

$$U \cap W = \begin{cases} x(t) < \phi_{m0} \\ x(t) > \phi_{m0} - \dot{x}(t)dt \\ \dot{x}(t) > 0 \\ y(t) \le y_C \end{cases} \qquad (47)$$

The probability that U and W occur simultaneously at time t is:

$$\begin{aligned} P(U \cap W) &= \int_{-\infty}^{y_C} \int_{0}^{\infty} \int_{a-\dot{x}dt}^{\phi_{m0}} pdf(x, \dot{x}, y) dx d\dot{x} dy \\ &= dt \int_{-\infty}^{\phi_{m0}} \int_{0}^{\infty} \dot{x} \cdot pdf\left(x = \phi_{m0}, \dot{x}, y \right) d\dot{x} dy \end{aligned} \qquad (48)$$

By definition, the *CDF* is:

$$CDF\left(y_C \mid \phi_{m0} \right) = \frac{P(W \cap U)}{P(U)} = \frac{\int_{-\infty}^{\phi_{m0}} \int_{0}^{\infty} \dot{x} \cdot pdf\left(x = \phi_{m0}, \dot{x}, y \right) d\dot{x} dy}{\int_{0}^{\infty} \dot{x} \cdot pdf\left(x = \phi_{m0}, \dot{x} \right) d\dot{x}} \qquad (49)$$

Differentiation of (49) yields a *pdf* of the value of the dependent process at the instant of upcrossing:

$$pdf(y_C) = \frac{\int_0^\infty \dot{x} \cdot pdf(x = \phi_{m0}, \dot{x}, y_C) d\dot{x}}{\int_0^\infty \dot{x} \cdot pdf(x = \phi_{m0}, \dot{x}) d\dot{x}} \qquad (50)$$

For the considered case of a dynamical system with piecewise linear term, all of the processes are normal and their mutual dependence is completely described by the appropriate covariance moments. This information is available as all of these processes are presented by Fourier series. The integrals in equation (50) can be evaluated symbolically:

$$pdf(y_C) = \frac{\sigma_{\dot{x}|xy}}{\sqrt{2\pi}} \exp\left(-\frac{\left(m_{\dot{x}|xy}(y_C)\right)^2}{2\sigma^2_{\dot{x}|xy}} \right) + \frac{m_{\dot{x}|xy}(y_C)}{2} \left(1 - \text{erf}\left(-\frac{m_{\dot{x}|xy}(y_C)}{\sqrt{2}\sigma_{\dot{x}|xy}} \right) \right) \qquad (51)$$

- $m_{\dot{x}|xy}$ and $\sigma_{\dot{x}|xy}$ are the conditional mean and the conditional standard deviation of the derivative of the process *x*(*t*) if the processes *x*(*t*) and *y*(*t*) took particular values. Note that the conditional mean is a function of the value of the process *y* at upcrossing, while the standard deviation is a constant; erf is the standard error function (see Belenky *et al.* (2013) for details).

The probability of capsizing after an upcrossing event is expressed as:

$$P(y_C < 0) = \int_{-\infty}^{0} pdf(y_C) dy_C \qquad (52)$$

Equation (52) completes the solution of the rare problem.

4.5 Probability of Rare Transitions

The final result of the rate of transitions (capsizes) can only be resolved using quadratures:

$$\xi_C = \frac{1}{2\pi} \frac{\sigma_{\dot{x}}}{\sigma_x} \exp\left(-\frac{\phi^2_{m0}}{2\sigma^2_x} \right) \int_{-\infty}^{0} f_C(y_C) dy_C \qquad (53)$$

- σ_x and $\sigma_{\dot{x}}$ are the standard deviation of process *x* and its derivative.

Most of the basic theory of rare transitions with random stiffness was published in Belenky *et al.* (2010, 2011). The assumption of the independence of the process *x* and its derivative, which appeared in those publications, was abolished in order to obtain a more general solution (Belenky *et al.*, 2013, 2013a). The latter works also contain a closed-form solution to (51), which was not available in the earlier publications.

The main outcome of the basic study of rare transitions in dynamical system with random stiffness is that the critical roll rate becomes time variant and random as well. As part of this study, the distribution of the value of dependent process at the instant of upcrossing was derived.

5. NUMERICAL EXTENSION OF BASIC THEORY OF RARE RANDOM TRANSITIONS WITH RANDOM STIFFNESS

5.1 Towards a Time-Domain Solution

A comparison of a basic theory of random transition (Section 2) with its numerical extension (Section 3) shows a commonality in their approaches but some significant differences in technique. The understanding that numerical techniques may be quite different from the solution for the simpler piecewise linear system came from a number of studies, where "theoretical" methods were attempted in a more direct way.

The calculation of instantaneous GZ curve in waves was described in Belenky and Weems (2008). Belenky *et al.* (2010) describe a method of tracking the maximum of the GZ curve in waves. However, an attempt to use the theoretical formula for upcrossing rate, as proposed in Belenky *et al.* (2008), showed a significant discrepancy from statistical estimates for stern quartering seas. This discrepancy was not observed in beam seas.

The reason for this discrepancy is the dependence between roll angles and roll rate in stern quartering seas (Belenky and Weems, 2012; Belenky *et al.*, 2013). The roll angles and rates are not correlated (see, for example, Bendat and Piersol, 1986). However, the absence of correlation means independence only for a normal process. Since large-amplitude roll motions may be not normal, independence cannot be assumed based on an absence of correlation. In this case, dependence can be characterized through the joint moments of higher order, say the fourth joint moment (covariance is the second joint moment). It is possible that the dependence is somehow related with stability variation in stern quartering seas, as it was not observed in beam seas.

The probabilistic properties of the elements of GZ curve in waves was found to be quite complex (Belenky and Weems, 2008). As a result, the modeling of a threshold distribution is difficult. Difficulty is further increased by the necessity to include all dependencies in order to get the joint distribution required in formula (50). Even if such a distribution fit is proposed, it may be reasonable only near the mean value, while the distribution needs to be evaluated on the tail. It was concluded that this approach did not offer a practical solution.

The difficulties fundamentally originated from the necessity to model tails of multi-dimensional distributions. These distributions are needed for characterizing the values at the instant of upcrossing. Why not get this information directly from the simulated data?

This simple question has led to understanding that numerical methods may be quite specific and should be based on direct data analysis, *i.e.* statistics. This motivated a revision of the original work on the evaluation of probability of capsizing in beam seas. The description in Section 3 reflects the authors' current understanding of how the problem should be handled, which has evolved significantly since its first publication in Belenky *et al.* (2008).

5.2 Numerical Non-Rare Problem

The formulation of the non-rare problem for the case of time-dependent stiffness is almost identical to the case of constant stiffness. However, the requirement of the independence of upcrossings can be removed. Dependent crossing events will be addressed as a part of rare problem as will be described in the next subsection. As a result, the choice of the intermediate threshold becomes a matter of computational efficiency only.

5.3 Numerical Rare Problem

The solution for a dynamical system with random piecewise stiffness has shown that the critical roll rate depends on time. To account for the stability changes in wave in the numerical case, the critical roll rate is calculated at each upcrossing, and the effect of the wave is included in the rare simulations. The calculations start from the instant of upcrossing; and roll rate is perturbed until capsizing is reached; see Figure 8.

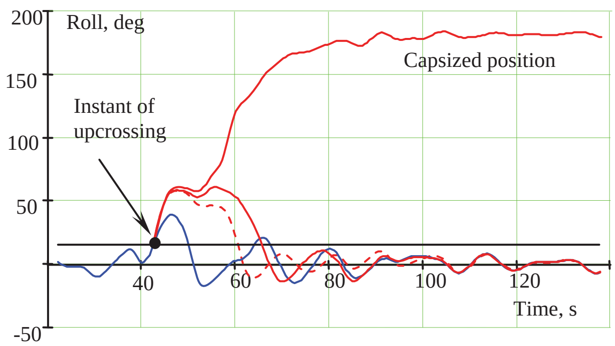

Figure 8: Calculation of critical roll rate

A detailed discussion of this algorithm, further referred as the Motion Perturbation Method (MPM), can be found in Spyrou *et al.* (2014). A particularly important point is how MPM is related to the definition of motion stability and the classic definition of ship stability given by Euler (1749). The result of the MPM calculation is a value of the metric of likelihood of capsizing (Belenky *et al.*, 2014):

$$y_i = 1 - \dot{\phi}_{Ui} + \dot{\phi}_{Cri} \; ; \quad i = 1, \ldots, N_U \qquad (54)$$

- $\dot{\phi}_{Crj}$ is the critical roll rate calculated for the i^{th} upcrossing, and $\dot{\phi}_{Ui}$ is the roll rate observed at the i^{th} upcrossing.

The next step should be the GPD extrapolation of the metric y to find the probability of exceeding 1, which is the value associated with capsizing per equation (54). However, GPD requires independent data and the independence of upcrossings is no longer required when choosing the intermediate threshold. To resolve this, the dependence or independence of successive upcrossings must be determined.

If capsizing does not occur, the perturbed time history returns to the unperturbed state after some time ("time of convergence"); see Figures 9. If the next upcrossing occurs within this time of convergence, it is considered to be dependent. If the effect of perturbation has no further influence, the next upcrossing is considered to be independent; see Figure 9. The critical roll rate is calculated for all upcrossings, but only the largest value in each set of dependent upcrossings is retained for further processing; see Spyrou *et al.* (2014) for further details.

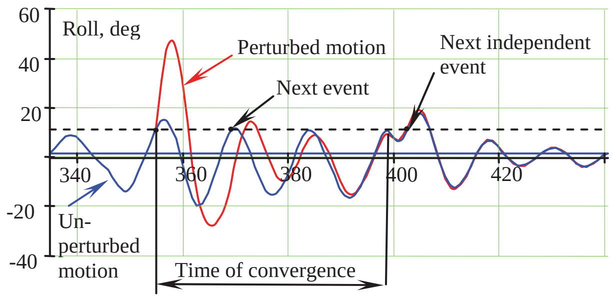

Figure 9: Dependent and independent upcrossings

The procedure for GPD extrapolation is similar to the constant stiffness case that was reviewed in Section 3.3. The only difference is that metric y is extrapolated and the target value associated with capsizing equals 1.

5.4 Initial Validation

As with any analytical method, validation is required if the method is to be put to practical use. But what would constitute validation of the split-time method of probability of capsizing? The split-time method is essentially the method of extrapolation; it is intended to evaluate the probability of capsizing based on a limited simulation data set. Thus, the validation of extrapolation is the more general question.

A statistical extrapolation method can be considered valid if its prediction is identical to value directly estimated from a sample. To do this, the sample must be large enough to support the estimation of the predicted event. To estimate the probability of capsizing, a sample must contain a number of capsizes so that the rate of capsizing can be estimated by direct counting. The extrapolation can then be applied to one or more small sub-samples of the data, each of which may or may not contain any capsizing events. If the estimates from the extrapolations and direct counting match, then the extrapolation method is valid.

While this idea seems straightforward, many issues need to be resolved to create a procedure of extrapolation. The development of this procedure is described by Smith and Campbell (2013), Smith (2014), and Smith *et al.* (2014) and summarized in Smith and Zuzick (2015). A key idea of the procedure is that the validation must be repeated systematically for the same condition in order to verify the confidence interval as well as for different speed, heading and wave environment in order to verify the robustness of the method.

Another question is how to get a sample that is large enough to capture such a rare event while retaining the essential nonlinear physics? For realistic wave conditions, millions of hours of simulation may be required to see capsizing. A particular problem is how to model stability variations in waves that play the central role in capsizing caused by pure loss of stability. ODE solvers may be fast enough to provided required simulation time, but ODE models may be quite questionable in terms of reproducing the stability variation.

For the present study, simulations were made with a 3 degree-of-freedom (heave, roll, pitch) time-domain simulation code which incorporates a novel volume-based calculation of the body-nonlinear Froude-Krylov and hydrostatic pressure forces. The algorithm is almost as fast as an ODE solver, but it captures the key features of the nonlinear wave forcing and restoring, allowing large, realistic irregular sea motion data sets to be generated. Description of the algorithm, implementation and verification is available from Weems and Wundrow (2013) and Weems and Belenky (2015).

The code was used to generate 1,000,000 hours of motion data for the ONR Topsides Tumblehome hull in random realizations of large, irregular stern quartering seas. 157 capsizes were observed, which allows a “true” value to be estimated. The split-time method was applied to 50 different sub-sets of the data, each of which consisted of 100 hours of data. The observed and extrapolated capsizing rates, with confidence intervals, are plotted in Figure 10. The percentage of successful extrapolations is 96%, which is very close to 95% of confidence probability. Details of the validation can be found in Belenky *et al.* (2014).

5.5 Summary on the Numerical Extension of the Basic Theory for Random Stiffness

Numerical extension of the basic theory of random transition in a dynamical system with random stiffness was initially published in Belenky *et al.* (2013), with an exponential distribution as a model for the tail of the metric. Subsequent publications (Belenky *et al.*, 2014; Spyrou *et al.*, 2014) include the switch to GPD, which has led to a successful initial validation.

The development of this numerical extension can be summarized as follows:

- It is possible to estimate the probability of capsizing numerically without any assumption on roll stiffness
- The problem can be solved by GPD extrapolation of the metric of likelihood of capsizing (equation 54)
- The metric fully accounts for the nonlinearity of dynamical system; it contains a critical roll rate computed by perturbations
- The metric can be seen as an implementation of both the classical definition of ship stability and the general definition of motion stability
- Roll rate and angles may be dependent in stern quartering seas, while remaining uncorrelated.

The method has been successfully tested using a large volume of ship motion generated with a volume-based 3-DOF simulation code. This test provides a very promising but limited validation of the method, as the results of the 3-DOF simulation should be considered to be a qualitatively rather than a quantitatively correct representation of ship capsizing. The numerical extension cannot be considered complete until it has been fully implemented and validated with a more complete time-domain simulation code such as LAMP.

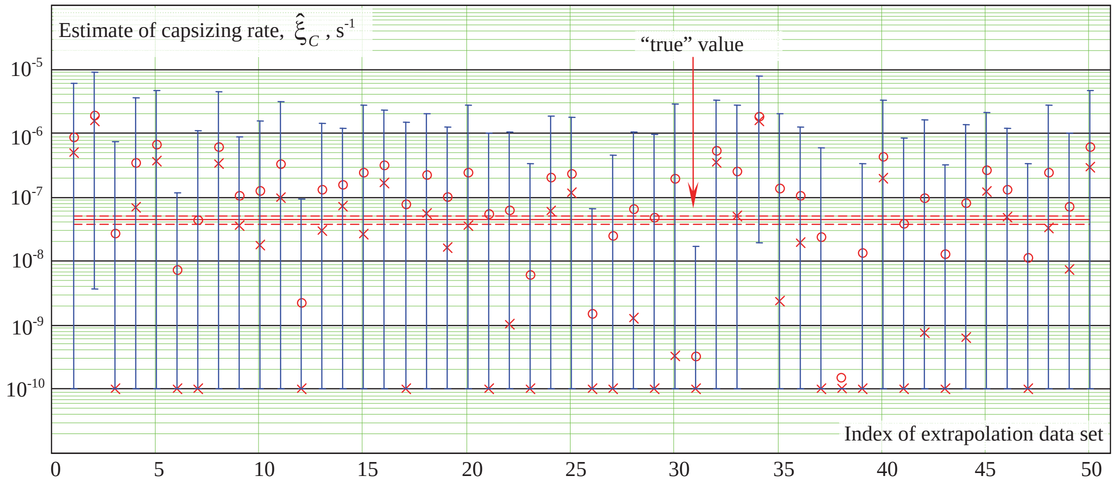

Figure 10: Results of initial validation performed for 50 validation data sets for ONR tumblehome

6. CONCLUSIONS AND FUTURE WORK

This paper has reviewed work under the ONR project "A Probabilistic Procedure for Evaluating the Dynamic Stability and Capsizing of Naval Vessels". The review was limited to the results related to pure loss of stability, without consideration of effects from wind, surge, sway, or yaw. The main focus of the study was on the effect of random stability variations in waves. The research was reviewed in four following steps:

- Basic theory of rare random transitions
- Numerical extension of the basic theory of rare random transition
- Basic theory of rare random transition in a dynamical system with random stiffness
- Numerical extension of the theory of rare random transition in a dynamical system with random stiffness.

The result of the study is a procedure of physics-based statistical extrapolation using a limited data set from nonlinear time-domain numerical simulation. The procedure consists of the following steps:

- Prepare an extrapolation data set of simulation data; typically about 100 hours of total duration and consisting of a number of records approximately 30 minutes each
- Set an intermediate threshold providing a reasonable number (thousands) of upcrossings to be observed
- For each upcrossing, use perturbation simulations to find the critical roll rate leading to capsize, and then use the difference between the observed and critical roll rate to calculate the value of the metric of the likelihood of capsizing; then remove dependent data from the dataset
- Fit GPD with the metric data; evaluate the estimate of the capsizing rate and its confidence interval.

So far, this procedure has had a very limited validation for one condition and has not been fully implemented with LAMP or other advanced hydrodynamic code. The following are the next steps in the development of the method:

- Bring the validation to reasonable completion by considering more conditions
- Address implementation issues related with the consideration of 6 DOF in the solution of the non-rare and rare problems
- Consider the inclusion of hydrodynamic diffraction and radiation forces in the solution of the rare problem.

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