Analytical Study of the Capsize Probability of a Frigate

Frédéric Le Pivert, *Research Institute of the French Naval Academy, Brest, France*
frederic.le pivert@ecole-navale.fr

Abdelkader Tizaoui, *Institut Polytechnique des Sciences avancées, DRI, Paris, France*

Radjesvarane Alexandre, *ENSAM, Paristech, Paris, France*

ean-Yves Billard, *Research Institute of the French Naval Academy, Brest, France*

ABSTRACT

The roll motion for a ship in a transverse sea can be represented by a one degree of freedom model. Equations are derived to write analitycally the probability density function of roll angle, roll speed and roll excitation moment. Also a capsize cirterium is shown to have a whole process for a capsize probability calculation.

Keywords: roll motion, capsizing, Fokker-Planck-Kolmogorov equation, characteristic method.

1 INTRODUCTION

Ship stability is one of nowadays worries. The old criteria that were defined in the first part of the 20th century are based on static sta-bility. The new generation of criteria should be based on dynamical stability. These criteria are not simple evaluation but should be also quan-tification of the risks for a ship on any sea.

All ships are different but their behaviours on sea could be modelled with the same type of equations. Here the problem is limited to a ship sailing in transverse sea. It is assumed that in this case, roll motion could be represented by a one degree of freedom (1-DOF) model. So the goal of the paper is to solve the Fokker-Plank-Kolmogorov equation (FPK) associated to the dynamical problem and to obtain the probabil-ity density function of the three following vari-ables: roll angle, roll speed and roll excitation. The solution obtained has been tested on realis-tic situation. A capsize criterium is also added to have a full way to obtain a capsize probability.

Several methods have already been suggested to estimate large roll angles and stability fail-ures. The Peak Over Threshold Method (Mc-Taggart 2000) and Envelope Peak Over Thresh-old Method (Belenky & Campbell 2011) use statistical extrapolation on relatively small am-plitudes to find the largest motions probabil-ity (Campbell 2014). The extrapolation tech-nique is also a real issue for roll motion proba-bility. One way is the split-time method. The split-time method divides the problem in two parts considering the ship behavior is different whether the roll angle is below or above a given threshold. The idea is to fit only the largest an-gles distribution (Belenky, 2014) and could be applicated for both Peak Over Threshold and Envelope Peak Over Threshold methods.

Melnikov methods have also been largely discussed in the 1990s and in the 2000s (Hsieh et al. 1994, Scolan, 1997, Jiang et al. 2000, McCue & Troesch 2005). Melnikov methods can deter-mine properly whether a sea state is dangerous or not. Markov methods use the dynamics of the system to find the complete expression of the roll motion probability by solving an FPK. The present paper uses one of these methods.

2 MATHEMATICAL ANALYSIS

2.1 Markov Methods

The aim of this group of methods is to consider roll motion as a Markovian process. In their paper Roberts and Vasta (Roberts & Vasta 2000) describe the time evolution of the energy of roll motion with a white noise as system perturbation.

More recent methods preserve the roll mo-tion equation and consider the perturbation as a filtered Gaussian white noise according to Spanos ARMA filters theory (Spanos 1983). This method was applied to uncoupled roll mo-tion by Francescutto and Naito (Francescutto & Naito 2004) and the method was fully developed by Su and Falzarano (Su & Falzarano 2011). This method overcomes the difficulty to deal with a noise which does not have any remark-able property. On the other hand the system becomes larger and new variables appear with-out any physical sense. By using this method, the FPK of the complete system for both old and new variables can be obtained. These previous authors derive numerically the equation.

In the present paper the FPK is derived analytically. This derivation needs in return some simplification of dynamics.

2.2 Derivation of the Methods

Consider the following adimensioned roll motion equation:

$$\ddot{\phi} + \lambda_1 \dot{\phi} + \lambda_2 \dot{\phi}|\dot{\phi}| + c(\phi) = f(t), \qquad (1)$$

where:

- φ is the roll angle,
- t is the time,
- $\dot{y}$ means the time derivative of the quantity y,
- λ_1 is the linear damping coefficient,
- λ_2 is the quadratic damping coefficient,
- c is the restoring moment,
- f is the external random moment.

The equation (1) is the expression of the principle of dynamics applied to roll motion. To obtain (1), all the moments where divided by $I_{xx}\omega_0^2$, where I_{xx} is the total inertia in roll of the ship and ω_0 is the natural roll frequency of the ship.

The term f is supposed to be a filtered white noise. The considered filter is defined by the following equation:

$$\ddot{Z} + V_1\dot{Z} + V_0 Z = \gamma W, \quad (2)$$

where V_1, V_0, γ are constant. W is a Gaussian white noise. so it leads to the following system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\lambda_1 x_2 - \lambda_2 x_2 |x_2| - c(x_1) + x_3, \\ \dot{x}_3 = x_4, \\ \dot{x}_4 = -V_1 x_4 - V_0 x_3 + \gamma W. \end{cases} \quad (3)$$

In (3), we have:

- $\varphi = x_1$,
- $\dot{\phi} = x_2$,
- $f = x_3$.

The system (3) is rewritten with vectors:

$$\dot{X} = F(X) + GW. \quad (4)$$

This can lead to the Fokker-Planck-Kolmogorov equation:

$$\partial_t P = -\nabla.(P\,F) + \frac{\gamma^2}{2}\partial_{x_4} P, \quad (5)$$

where P is the probability density function of the random variable X.

Remark: According to (Francescutto & Naito 2004), a 4^{th} order-filter at least should be ap-plied. The present filter has a smaller order filter only to write an analytical formula for P . Here is one simplification to get the formula.

To integrate (5), it was chosen to take the space Fourier transform of this equation. Be-cause F_2 ($F_2 = -\lambda_1 x_2 - \lambda_2 x_2|x_2| - c(x_1) + x_3$) Fourier transform has no analytical expression, new hypotheses should be made:

- linearization around equilibrium $c(x_1) = c_i(x_1 - x_{eq})$,
- damping linearization, $\lambda_2 = 0$,

where c_i is the restoring coefficient around a con-sidered point of equilibrium x_{eq}, x_{eq} is defined by $c(x_{eq}) = 0$. Here it is chosen to derive (5) assuming the hypotheses. It is assumed a boat has three heel angles of equilibrium on each side: $\varphi = 0$, $\varphi = \pm\varphi_V$, $\varphi = \pm\pi$. Choosing a lineariza-tion around these points is considering $c(x_1)$ as a 5^{th} order polynom. The form of c is given by (6):

$$c(x_1) = C\,x_1(x_1^2 - \varphi_V^2)(x_1^2 - \pi^2). \quad (6)$$

This method is equivalent to the piece-wise linearization method (Belenky 1993). The dif-ference is: in the present method the roll mo-tion is supposed to be fully forced by the exter-nal moment, whereas in (Belenky 1993) the roll angle is considered as a solution of (1). Using that method, a transition solution calculated by considering f = 0 should be taken into account.

The Fourier transform of (5) is:

$$\partial_t \hat{P} = \Big(c_i\xi_2\partial_{\xi_1} + (\xi_1 - \lambda_1\xi_2)\partial_{\xi_2} + (\xi_2 - V_0\xi_4)\partial_{\xi_3} + (\xi_3 - V_1\xi_4)\partial_{\xi_4}\Big)\hat{P} + \frac{\gamma^2}{2}\xi_4^2\hat{P} \quad (7)$$

The equation (7) is a transport equation. So if we know an integrable solution $\hat{P}_0(\xi_1, \xi_2, \xi_3, \xi_4)$ at t = 0, there exists an integrable solu-tion $\hat{P}(t, \xi_1, \xi_2, \xi_3, \xi_4) = \hat{P}_t(\xi_1, \xi_2, \xi_3, \xi_4)$ with the same measure at every t > 0. Fourier transform make this property true for P solution of (5). Here to obtain (7) $\hat{P}$ has been supposed square-integrable. So P is square-integrable.

A method of characteristics is ap-plied: characteristic curves are curves

$s \rightarrow (t(s), \xi_1(s), \xi_2(s), \xi_3(s), \xi_4(s))$ who check the following condition:

$$\begin{cases} \frac{dt}{ds} = 1, \\ \frac{d\xi_1}{ds} = c_i\xi_2, \\ \frac{d\xi_2}{ds} = \lambda_1\xi_2 - \xi_1, \\ \frac{d\xi_3}{ds} = V_0\xi_4 - \xi_2, \\ \frac{d\xi_4}{ds} = V_1\xi_4 - \xi_3. \end{cases} \quad (8)$$

So t = s and

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \\ \dot{\xi}_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & c_i & 0 & 0 \\ -1 & \lambda_1 & 0 & 0 \\ 0 & 1 & 0 & V_0 \\ 0 & 0 & -1 & V_1 \end{pmatrix}}_{A} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} \quad (9)$$

The solution of (9) is written this way:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \sum_{k=1}^{4} \kappa_k W_k e^{w_k s}, \quad (10)$$

where w_k are the eigenvalues of A, W_k are the eigenvectors of A and κ_k are determined by ini-tial conditions.

Remark: Now the choice for a second order filter can be justified. If a larger order filter had been chosen, this would have lead to a larger ma-trix A. Then it becomes impossible to calculate analytically the eigenvalues and the eigenvectors of A.

$$\begin{aligned} w_1 &= \frac{\lambda_1 + \sqrt{\lambda_1^2 - 4c_i}}{2}, \\ w_2 &= \frac{\lambda_1 - \sqrt{\lambda_1^2 - 4c_i}}{2}, \\ w_3 &= \frac{V_1 + \sqrt{V_1^2 - 4V_0}}{2}, \\ w_4 &= \frac{V_1 - \sqrt{V_1^2 - 4V_0}}{2}, \end{aligned} \quad (11)$$

and

$$\mathbf{W}_1 = \begin{pmatrix} -w_2(w_1(\lambda_1 - V_1) - c_i + V_0) \\ w_1(\lambda_1 - V_1) - c_i + V_0) \\ V_1 - w_1 \\ 1 \end{pmatrix}, \quad (12)$$

$$\mathbf{W}_2 = \begin{pmatrix} -w_1(w_2(\lambda_1 - V_1) - c_i + V_0) \\ w_2(\lambda_1 - V_1) - c_i + V_0) \\ V_1 - w_2 \\ 1 \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 0 \\ 0 \\ w_3 \\ 1 \end{pmatrix}, W_4 = \begin{pmatrix} 0 \\ 0 \\ w_4 \\ 1 \end{pmatrix} \quad (13)$$

After calculation, a formula-tion of $\xi_1(s), \xi_2(s), \xi_3(s), \xi_4(s)$ using $\xi_1(0), \xi_2(0), \xi_3(0), \xi_4(0)$ is obtained. Along the characteristic curves, it can be written:

$$\frac{d\hat{P}}{ds} = \frac{\partial \hat{P}}{\partial t}\frac{dt}{ds} + \frac{\partial \hat{P}}{\partial \xi_1}\frac{d\xi_1}{ds} + \frac{\partial \hat{P}}{\partial \xi_2}\frac{d\xi_2}{ds} + \frac{\partial \hat{P}}{\partial \xi_3}\frac{d\xi_3}{ds} + \frac{\partial \hat{P}}{\partial \xi_4}\frac{d\xi_4}{ds} = -\frac{\gamma^2}{2}\xi_4^2\hat{P} \quad (14)$$

The only solution is:

$$\hat{P}(s) = \hat{P}(0) \exp\left(-\frac{\gamma^2}{2}\int_0^s \xi_4^2(u)du\right) \quad (15)$$

As the expression $\int_0^s \xi_4^2(u)du$ is difficult to un-derstand written in this way, ξ_i are replaced by $\zeta_i = \kappa_i e^{w_i s}$.

So it leads to:

$$\xi_4 = \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4. \quad (16)$$

Let express the solution as a function of t:

$$\hat{P}_t \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \hat{P}_0 \begin{pmatrix} \xi_1(0) \\ \xi_2(0) \\ \xi_3(0) \\ \xi_4(0) \end{pmatrix} .e^{\Psi} \quad (17)$$

where $\xi_1(0)$, $\xi_2(0)$, $\xi_3(0)$, $\xi_4(0)$ could be expressed with ξ_1, ξ_2, ξ_3, ξ_4 and t.

$$\Psi = -\frac{\gamma^2}{2}\left(\frac{\zeta_1^2}{2w_1} + \frac{\zeta_2^2}{2w_2} + \frac{\zeta_3^2}{2w_3} + \frac{\zeta_4^2}{2w_4} + \frac{2\zeta_1\zeta_2}{w_1+w_2} + \frac{2\zeta_1\zeta_3}{w_1+w_3} + \frac{2\zeta_1\zeta_4}{w_1+w_4} + \frac{2\zeta_2\zeta_3}{w_2+w_3} + \frac{2\zeta_2\zeta_4}{w_2+w_4} + \frac{2\zeta_3\zeta_4}{w_3+w_4} \right) \quad (18)$$

With (17) a complete formula for the time depending probability density function of the random variables ξ_1, ξ_2, ξ_3, ξ_4 is given at every time t. The function Ψ depends on the random vari-ables (18) and describes an ellipse. The term $\hat{P}_0$ is a displacement of the properties at t = 0 along the characteristic curves.

So the solution of (5) is:

$$P_t \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = P_0 \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} * \chi \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad (19)$$

where $x_1(0)$, $x_2(0)$, $x_3(0)$, $x_4(0)$ can be expressed with x_1, x_2, x_3, x_4 and t and $*$ means the con-volution product. The function χ is the in-verse Fourier transform e^{Ψ} and is a gaussian probability law for random variables y_1, y_2, y_3, y_4 which are derived from x_1, x_2, x_3, x_4 with an au-tomorphism. The following initial condition are apllied:

$$P_0 \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \delta \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad (20)$$

where δ is the standard Dirac distributon.

So the final solution of (5) is:

$$P_t \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \chi \begin{pmatrix} x_1 - x_1(0) \\ x_2 - x_2(0) \\ x_3 - x_3(0) \\ x_4 - x_4(0) \end{pmatrix} \quad (21)$$

In this way, the solution does not depend on time.

The solution has a gaussian form for the 4 variables. This result is in accordance with (Be-lenky 1993) considering only forced oscillations.

3 RESULTS

3.1 Method

All results were obtained by generating a large number of simulations in which the sea state remain the same. The software used for the simulation is FREDYN. FREDYN calculate the 6-DOF dynamics of a given boat with the potential flow assumption. Here the boat used for simulation is the F70-frigate of the French Navy. The case tested is the frigate in trans-verse sea with 0 or 6 knots forward speed.

3.2 Time-independance Verification

The time-independance of the probability is tested with long simulations. The hypotheses of calculation used in the first part are not taken into account.

For a serie of 50 simulations lasting 5 hours in a sea state defined by a Pierson-Moskowitz spectrum with a significant wave height H_S = 12.4m and a mean wave period T_P = 12.7s, the max-imum of the roll angle for each simulation have been situated in time during the simulation. The number of maxima occuring before a certain time is counted and represented in Figure 1. The frigate's forward speed is 6 knots.

time of the maximum	number of maxima
< 2000s	8
< 4000s	11
< 6000s	23
< 8000s	27
< 10000s	31
< 12000s	33
< 14000s	37
< 16000s	46
< 18000s	50

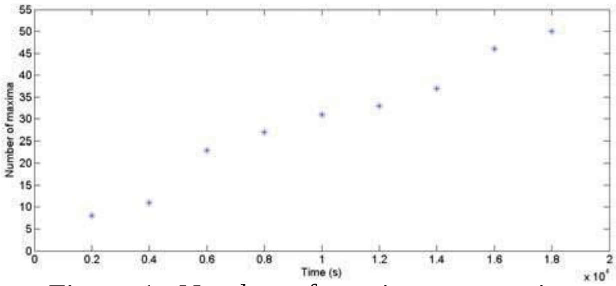

Figure 1: Number of maxima versus time

The number of maxima is linearly growing, so the probability associated to roll motion is time-independant.

3.3 Probability Density Function Esitimation

A direct estimation of the probability density function has been calculated for 0 forward speed frigate in 5 sea states. The estimation is made according that over a long time the probability is

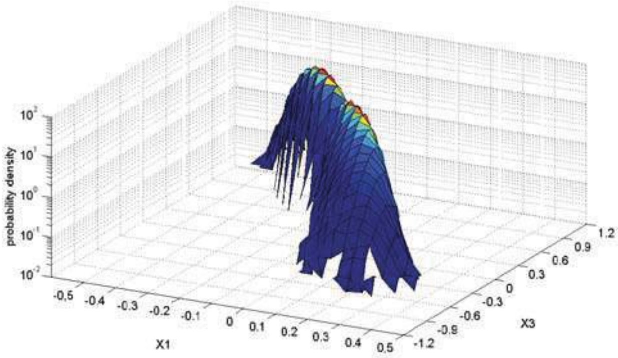

Figure 2: Conditional probability density function $P(x_1, x_2 = 0|x_3)$

stable. The first results showed a similar shape for the probability density function between sea states. To compare the sea states, Bayes' for-mula was used so:

$$P(x_1, x_2, x_3) = P(x_1, x_2|x_3).P(x_3), \qquad (22)$$

where | means knowing. The conditonal probability is supposed to be normal. For representa-tion, the results are taken at $x_2 = 0$. Figure 2 show a prefered axis of the (x_1, x_3)-plan. A for-mula is given with variable change: $Y_1 = kx_1+x_3$ and $Y_2 = kx_3 - x_1$. The variable Y_2 is describ-ing the evolution along the axis and Y_1 describes the evolution perpendicularly to the axis. The formula for $P(x_1, x_2 = 0|x_3)$ is of the following form:

$$P(x_1, x_2 = 0|x_3) = \exp\left(-\begin{pmatrix} h_0 + h_1Y_1 + h_2Y_1^2 \\ +g_1Y_2 + g_2Y_2^2 \\ +g_3Y_2^3 + g_4Y_2^4 \end{pmatrix}\right). \qquad (23)$$

The parameters k, h_0, h_1, h_2, g_1, g_2, g_3, g_4 are calculated for 5 different sea states and the re-sults are written in the following table.

H_S	T_P	h_0	h_1	h_2	k
9.270m	12.36s	-4.595	16.26	418.0	-0.4058
9.465m	12.57s	-5.981	15.96	422.6	-0.4141
9.660m	12.79s	-3.557	17.08	395.7	-0.4238
9.758m	12.90s	-3.963	15.68	418.6	-0.4256
10.448m	12.40s	2.994	16.96	401.5	-0.4140

H_S	T_P	g_1	g_2	g_3	g_4
9.270m	12.36s	0.8710	-10.680	-6.489	41.45
9.465m	12.57s	0.6514	-9.714	-4.792	34.70
9.660m	12.79s	0.5147	-9.380	-3.882	31.26
9.758m	12.90s	0.5350	-9.441	-3.870	31.63
10.448m	12.40s	0.1893	-2.475	-1.229	7.584

The parameters h_1, h_2, k have really similar values for all the sea state. For the other parameters, the last sea state (H_S = 10.448m, T_P = 12.40s) gives values contrastive in the other sea states. The significative wave height of the last sea state is significantly higher than the others and, in the same time, the mean wave period remains the same. This leads to a much more dangerous sea state and ex-plains why this sea state is associated to con-strative values for h_0, g_1, g_2, g_3, g_4. Nonetheless the value of h_0, g_1, g_2, g_3, g_4 are of close order. This could indicate a slight evolution of these parameters with the sea state. The similar val-ues for h_1, h_2, k indicate these parameters are almost constant. The parameter k gives the di-rection of the prefered axis at $x_2 = 0$ and h_1, h_2 the decrease of the probability for points of the (x_1, x_3)-plan which are not on the axis.

The form of the results obtained by numerical simulation is in accordance with the analytical developpement such as the gaussian probability law for the random variable Y_1.

3.4 Capsizing Criteria

The goal of the probability density estimation is to obtain a capsize probability. This leads to a

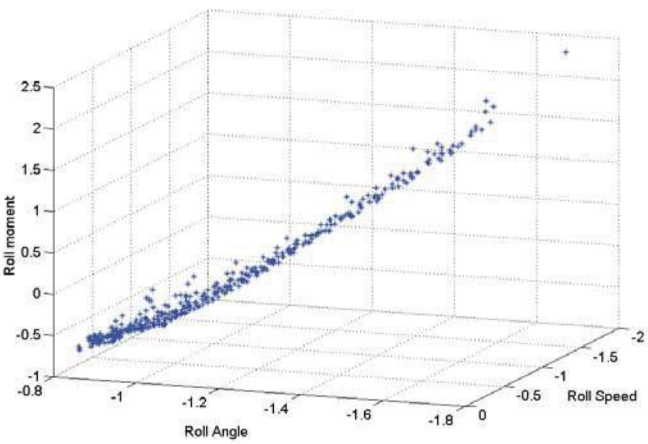

Figure 3: points of the angle-speed-moment-space with $\varphi > 48^\circ$, $\varphi.\dot{\varphi} > 0$

search for criteria of capsizing.

Here are compared simulations in which the roll angle has been really large and simulations leading to capsize. The sea state is still defined by a Pierson-Moskowitz spectrum, H_S = 12.5m and T_P = 12.6s. In the space defined by an-gle, speed and moment (x_1, x_2, x_3), points cor-responding to angle over 50° and speed of the same sign like angle (situations getting closer to capsize) are extracted Figure 3.

The points seem to get aligned in a same plan. Figure 4 show the points in this plan defined by two arbitrary variables V_1, V_2. Red points correspond to simulations getting to capsize and blue points correspond to simulations without capsizing.

A stable area can be defined with the trajec-tories which do not lead to capsize. So even for a large angle, the boat could escape such danger-ous situation. Then the probability of capsizing is the probability for the trajectory in the angle-speed-moment space to come out of the stable domain.

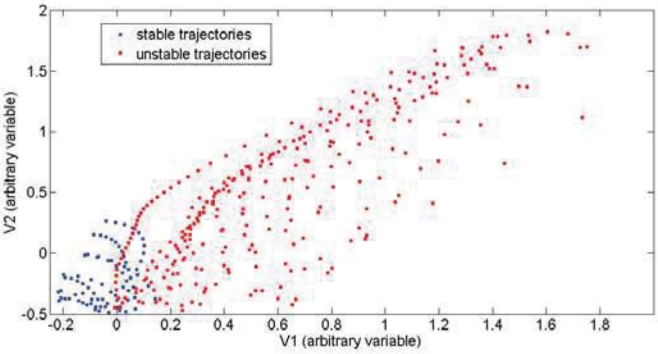

Figure 4: comparison of trajectories either stable or leading to capsize

4 CONCLUSION

In the paper an analytical formula of the probability density function of the linear roll motion has been obtained. The properties have been tested for simulations with realistic sea state. The results are partially in accordance, but the time independance remains exact both for linear and non-linear roll motion. In some case, the gaussian law proved for the linear motion remains exact for the non-linear motion. On top of that capsize criterium has been found for the calculation of a capsize probability with the probability density function of the roll angle, speed and moment.

5 REFERENCE

McTaggart, K.A., 2000, Ship Capsize risk in a Seaway Using Fitted Distributions to Roll Maxima, ASME Journal of Offshore Mechanics and Artic Engineering, Vol. 122, No. 2, pp.141-146.

Belenky, V. and Campbell, B.L., 2011, Evaluation of the Exceedance Rate of a Stationary Stochastic Process by Statistical Extrapolation Using the Envelope Peaks over Threshold (EPOT) Method, Hydrodynamics Department Report, NSWCCD-50-TR-2011/032, Naval Surface Warfare Center Carderock Division, MD.

Hsieh, S.-R., et al., 1994, A Nonlinear Probabilistic Method for Predicting Vessel Capsizing in Random Beam Seas, Proceedings: Mathematical and Physical Sciences, Vol. 446, No. 1926, pp.195-211.

Scolan, Y.-M., 1997, Technical Note on ship rolling associated to high degree polynomial moment using Melnikov methods, Applied Ocean Research, Vol. 19, pp 225-234.

Jiang, C., et al., 2000, Capsize Criteria for Ship Models with Memory-Dependent Hydrodynamics and Random Excitation, Philosophical Transactions: Mathematical, Physical and Engineering Sciences, Vol. 358, No. 1771 The Nonlinear Dynamics of Ships, pp.1761-1791.

McCue, L. and Troesch, A., Probabilistic determination of critical wave height for a multi-degree of freedom capsize model, Ocean Engineering, Vol. 32, pp. 1608-1622.

Roberts, J.B., Vasta, M., 2000, Markov Mod-elling and Stochastic Identification for Nonlin-ear Ship Rolling in Random Waves, Philosoph-ical Transactions: Mathematical, Physical and engineering Sciences, Vol. 358, No. 1771 The Nonlinear Dynamics of Ships, pp. 1917-1941.

Spanos, P.-T.D., 1983, ARMA Algorithms for Ocean Wave Modelling, ASME Journal of Energy Resources Technology, Vol. 105, pp. 300-309.

Francescutto, A. and Naito, S., 2004, Large amplitude roll motion in a realistic seaway, International Shipbuilding Progress, Vol. 51, No. 2, pp. 221-235.

Su, Z. and Falzarano, J.M., 2011, Gaussian and non-Gaussian cumulant neglect application to large amplitude rolling in random waves, International Shipbuilding Progress, Vol. 58, No. 2, pp. 97-113.

Campbell, B., et al., 2014, On the Application of Generalized Pareto Distribution for Statistical Extrapolation in the Assesment of Dynamic Stability in Irregular Waves, 14th International Ship Stability Workshop, Kuala Lumpur, Malaysia.

Belenky, V., et al., 2014, Extrapolation and Validation Aspects of the Split-Time Method, 30th Symposium on Naval Hydrodynamics, Hobart, Tasmania, Australia.

Belenky, V., 1993, A Capsizing Probability Computation Method, Journal of Ship Research, Vol. 37, No. 3