PROBABILITY OF CAPSIZING IN BEAM SEAS WITH PIECEWISE LINEAR STOCHASTIC GZ CURVE

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ABSTRACT

The probability of capsizing for a dynamical system with time-varying piecewise linear stiffness is presented. The simplest case is considered, in which only the angle of the maximum of the restoring curve is changing. These changes are assumed to be dependent on wave excitation; such a system can be considered as a primitive model of a ship in beam seas, where all changes in stability are caused by heave motions. A split-time approach is used, in which capsizing is considered as a sequence of two random events: upcrossing through a certain threshold (non-rare problem) and capsizing after upcrossing (rare problem). To reflect the time-varying stability, a critical roll rate is introduced as a stochastic process defined at any instant of time. Capsizing is then associated with an upcrossing when the instantaneous roll rate exceeds the critical roll rate defined for the instant of upcrossing. A self-consistency check of the method, in which a statistical frequency of capsizing was obtained by time-domain evaluation of the response of the piecewise linear dynamical system and favorably compared with the theoretical prediction is described.

Keywords: *probability of capsizing, split-time method, piecewise-linear*.

1. INTRODUCTION

The calculation of the probability of capsizing for an intact ship in irregular seas is a formidable task, first of all because capsizing is an extremely rare event. Capsizing is also an extremely nonlinear phenomenon; it is, essentially, a transition from motion near a stable equilibrium to stable equilibrium. The combination of nonlinearity and rarity severely limits the set of available methods that can be applied to capsizing. A number of methods are available to treat rare events: in fact, the entire statistics of extremes is essentially focused on rarity (see for example Gumbel, 1962). These methods are based on the asymptotic properties of the tails of probability distributions and essentially rely on an extrapolation of the observed data.

The behavior of a nonlinear dynamical system under deterministic excitation has been addressed in a quite comprehensive manner (see for example Guckenheimer & Holms, 1983). The essence of the nonlinearity of roll motion is that the physics of the dynamical system changes with roll, leading to phenomena that are impossible for a linear system. The nonlinear qualities of roll motion have been the subject of study within the stability community for decades making, and an important part of program of the STAB conferences since 1975.

The change of the physical properties of the dynamical system makes the direct application of statistical extrapolation difficult as the probability may change along with the physics. This positions the probability of capsizing among the very special problems of stochastic dynamics — a very wide discipline stretching from mathematics (Arnold, 1998) to applied mechanics (cf. Roberts & Spanos, 2003).

As it is the combination of severe nonlinearity and extreme rarity that makes the problem so difficult, it seems natural to separate it in two problems and consider dynamics separately from probability. This approach was used by Themelis & Spyrou (2007) and Umeda, et al. (2007) for different scenarios of capsizing. The complex nonlinear behavior was considered with a sinusoidal wave or a deterministic wave group and probability of encounter was then evaluated using oceanographic statistics. Further development of this approach seems quite promising, especially with the consideration of its application with model tests (Bassler, et al., 2009).

The piecewise linear approach (Belenky, 1993; Paroka, et al., 2006; Paroka & Umeda 2006) represents another way to separate the problem. Since the change of physics presents the most significant problem, the separation of the problem comes at the points where the physics changes. These are the peaks of the roll righting arm (GZ) curve, which plays the role of the stiffness of the dynamical system. With even a linear approximation for the GZ curve

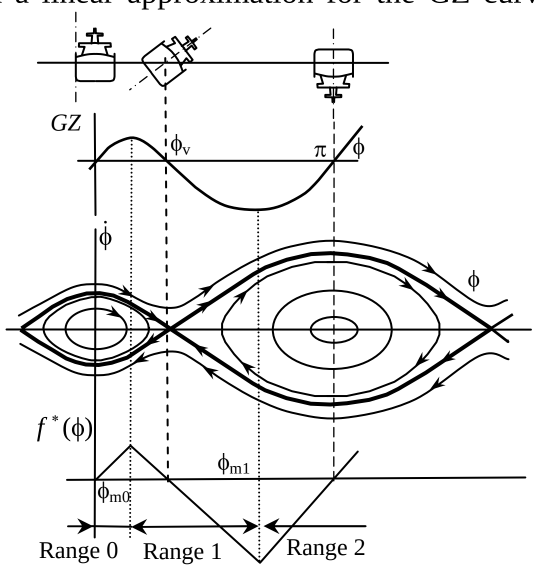

Figure 1. Phase plane topology and piecewise linear stiffness (Belenky, 1993).

in each range, it is possible to preserve the topology of phase plane (see Figure 1) and therefore to reflect the most important physical properties of roll motion.

Once the problem is separated, different solutions can be built for the first two ranges. The two solutions are connected at the separation point through initial conditions. The main advantage of the piecewise linear method for the problem of ship roll is that closed-form solutions can be presented for both ranges (Belenky, 1993). It should be noted, however, that any problem with a piecewise linear stiffness term does not necessarily have a closed-form solution.

While the ordinary differential equation for roll with piecewise linear stiffness has been shown to capture the key physical phenomena of nonlinear roll motion including capsizing, it is hardly a complete model for a ship rolling in waves. However, the piecewise linear system provides a theoretical model for studying nonlinear roll and building a bridge to a practical analysis method involving more complete models of roll.

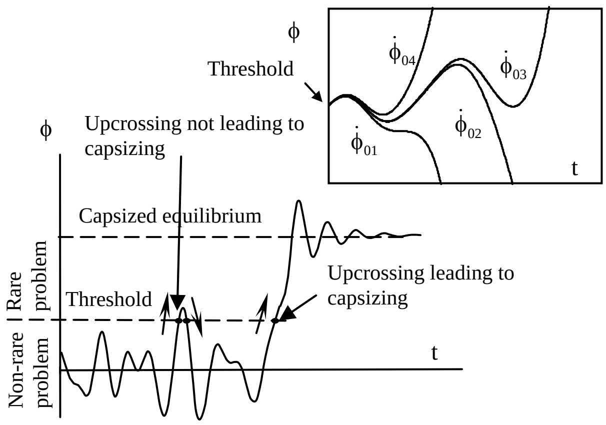

Figure 2. Summary of time-split method: separation principle and critical roll rate.

With the development of sophisticated methods and tools for numerical simulations of ship motions (Beck and Reed, 2001), the numerical analog of the piecewise linear method seems to be the logical next step. The split-time method described by Belenky, *et al.*

(2008) is just such a method. It is a direct generalization of piecewise linear method allowing the application of hydrodynamic simulation codes instead of linear ordinary differential equations. The key elements of method are illustrated in Figure 2.

As shown in Figure 2, capsizing is considered as an upcrossing of a threshold roll angle leading to the transition to another stable equilibrium. The first part of the method is the solution of a "non-rare" problem that provides enough roll motion data for the upcrossing probability to be evaluated. The second part is the solution of a "rare" problem, which consists of a series of short simulations starting at the threshold level, and is aimed at finding the "critical" roll rate at upcrossing that will lead to capsizing (see Figure 2 insert). The capsizing probability can then be evaluated as the probability of upcrossing with a roll rate exceeding the "critical" value.

In the initial development of the piecewise linear model of roll and its numerical application as the split-time method, the separation of the problem has assumed that the stiffness (GZ curve) was constant with time. The next step in the development of an approach is to consider changes of stiffness as the ship moves in waves. In order to provide a sound theoretical background for the development of this approach, it is first desirable to consider the solution of the basic piecewise linear method if the stiffness is random.

2. DYNAMICAL SYSTEM

Consider a dynamical system incorporating a piecewise linear restoring (stiffness) term with a random time-dependent decreasing part. The decreasing part, however, remains parallel to itself all the time, see Figure 3. Such a scheme is necessary to keep a linear relation and therefore normality throughout the problem. The stiffness term, indeed, is represented as a function of two variables, the roll angle and time; a surface illustrating such a function is shown in Figure 4.

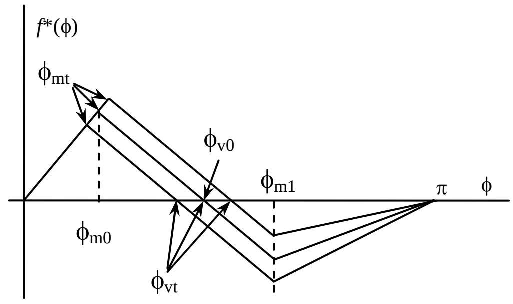

Figure 3. Piecewise linear stiffness term with time-varying decreasing part.

Values with zero in a subscript (*e.g.* ϕ_{m0}) are related to "calm-water" terms. The time-varying decreasing part of stiffness is assumed to be dependent on the heave motion only. While this model, as a whole, hardly describes the motion of a ship, even in beam seas, it nevertheless still possesses the key characteristics of the change of stability with ship motion.

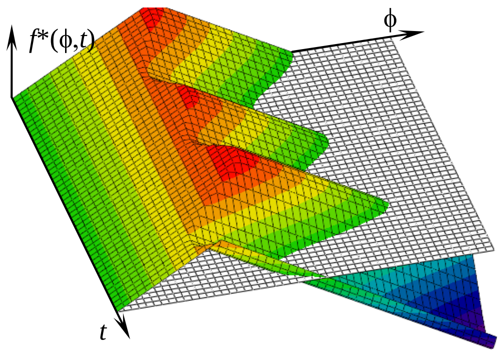

Figure 4. Piecewise linear stiffness term with time-varying decreasing part as a surface.

The value of angle of maximum of the GZ curve is assumed to have the following form.

$$\phi_m(t) = \frac{\phi_{m0}}{\omega_0^2}\left(k_d\left(\zeta(t)+d\right)+b_d\right). \tag{1}$$

Here ω_0 is natural frequency of roll in calm water, ϕ_{m0} is the initial boundary between two linear segments of the piecewise linear stiffness term (the angle of maximum of the GZ curve in calm water), d is the draft, ζ is the heave displacement, and k_d and b_d are linear fit coefficients, see Figure 5.

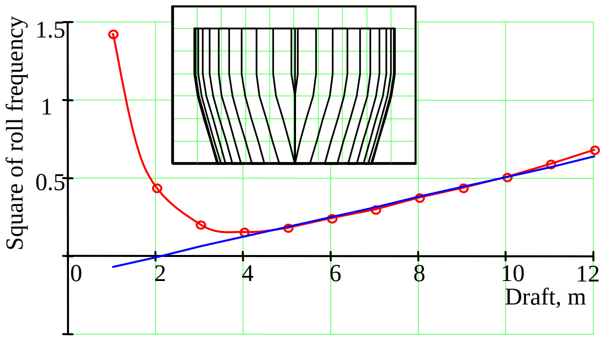

Figure 5. Dependence of square of natural frequency of roll on draft for a schematic ship (see the inset).

The entire piecewise linear stiffness term is expressed as:

$$f^*(\phi) = \omega_0^2 \begin{cases} \phi & |\phi| < \phi_m(t) \\ k_1\phi + b_1(t) & |\phi| \geq \phi_m(t) \end{cases}, \tag{2}$$

where linear coefficients k_1 and b_1 are calculated as follows:

$$k_1 = -\frac{\phi_{m0}}{\phi_{v0} - \phi_{m0}}$$
$$b_1(t) = -\phi_{v0} k_1 \left(1 + k_d \zeta(t)\right). \tag{3}$$

The dynamical system is described by two ordinary differential equations:

$$\ddot{\phi} + 2\delta_\phi \dot{\phi} + f^*(\phi) = f_{E\phi}(t)$$
$$\ddot{\zeta} + 2\delta_\zeta \dot{\zeta} + \omega_\zeta^2 \zeta = f_{E\zeta}(t). \tag{4}$$

Here $f_{E\phi}$ and $f_{E\zeta}$ are stationary ergodic stochastic processes describing wave excitation. The coupling between the equations is realized only through the stiffness term in the roll equation, and there is no influence of roll on heave.

3. NON-RARE PROBLEM

The objective of the non-rare problem is the evaluation of the probability of upcrossing the threshold roll angle, which is now changing with time. Below the threshold, the equations become decoupled and the solution for roll and heave are trivial:

$$\zeta(t) = \sum_{i=1}^{N_\omega} \zeta_{ai} \cos(\omega_i t + \gamma_{\zeta i} + \varphi_i)$$
$$\phi(t) = \sum_{i=1}^{N_\omega} \phi_{ai} \sin(\omega_i t + \gamma_{\phi i} + \varphi_i). \tag{5}$$

Here ω_i is the frequency of the components; ϕ_{ai} and ζ_{ai} are roll and heave component amplitudes; φ_i are random phases; and $\gamma_{\phi i}$ and $\gamma_{\zeta i}$ are roll and heave phase shifts respectively. These components can be calculated in the frequency domain without difficulty.

Similar expression can be written for the time history of the angle of the maximum of the GZ curve:

$$\phi_m(t) = \phi_{m0} + \sum_{i=1}^{N_\omega} \phi_{mai} \cos(\omega_i t + \gamma_{\zeta i} + \varphi_i)$$
$$\phi_{mai} = \frac{\phi_{m0}}{\omega_0^2} k_d \zeta_{ai}. \tag{6}$$

As the relation between the angle of the maximum and the heave motion is assumed to be deterministic and linear, the phases of the maximum are identical to those of heave motions. To identify the transition to rare problem, it is convenient to introduce a so-called "carrier" process representing an instantaneous difference between maximum and instantaneous roll angles:

$$x(t) = \phi(t) - \phi_m(t) + \phi_{m0}. \tag{7}$$

The transition to the rare problem can be now associated with the upcrossing of the level of the angle of maximum restoring in calm water by the carrier process. A time history of the carrier process can be expressed in a form of the Fourier series:

$$x(t) = \phi_{m0} + \sum_{i=1}^{N_\omega} x_{ai} \cos(\omega_i t + \gamma_{xi} + \varphi_i). \tag{8}$$

The amplitudes x_{ai} and phases γ_{xi} of the carrier process can be evaluated trivially as the roll and angle of maximum are presented with Fourier series containing the same frequency discertization:

$$x_{ai} = \sqrt{\phi_{mai}^2 + \phi_{ai}^2 + 2\phi_{mai}\phi_{ai}\sin(\gamma_{\phi i} - \gamma_{\zeta i})} \quad (9)$$

$$\gamma_{xi} = -\arctan\frac{\phi_{ai}\cos\gamma_{\phi i} - \phi_{mai}\sin\gamma_{\zeta i}}{\phi_{ai}\sin\gamma_{\phi i} + \phi_{mai}\cos\gamma_{\zeta i}}. \quad (10)$$

The carrier process $x(t)$ is differentiable; its derivative is expressed as:

$$\dot{x}(t) = -\sum_{i=1}^{N_\omega} x_{ai}\omega_i \sin(\omega_i t + \gamma_{xi} + \varphi_i). \quad (11)$$

The probability of upcrossing of the level ϕ_{m0} by the carrier process then can be calculated as (assuming applicability of Poisson flow):

$$P_U(T) = 1 - \exp(-\xi T). \quad (12)$$

With the upcrossing rate equal to:

$$\xi = \sqrt{\frac{V_{\dot{x}}}{V_x}} \exp\left(-\frac{\phi_{m0}^2}{2V_x}\right), \quad (13)$$

where the variance of the carrier, V_x, and its derivative, $V_{\dot{x}}$, can be derived from their Fourier presentations:

$$V_x = 0.5\sum_{i=1}^{N_\omega} x_{ai}^2\ ; \quad V_{\dot{x}} = 0.5\sum_{i=1}^{N_\omega} x_{ai}^2\omega_i^2 \quad (14)$$

4. RARE PROBLEM

4.1. Equation for Rare Problem

The substance of the rare problem is the characterization of the roll motion after an upcrossing occurs and the evaluation of the probability of capsizing after upcrossing.

Consider the roll equation (4) for the post-upcrossing range $|\phi| \geq \phi_m(t)$ after substitution of the time dependent stiffness (3):

$$\ddot{\phi} + 2\delta\dot{\phi} + k_1\phi = k_b\zeta(t) + k_1\phi_{v0} + f_{E\phi}(t)$$
$$k_b = k_1 k_d\ . \quad (15)$$

Equation (15) describes a dynamical system with a repelling force; following the usual assumption for the rare solution with a piecewise-linear system (Belenky, *et al.*, 2008), the influence of wave excitation is neglected. The influence of the time-variant stiffness expressed by the term $k_b\zeta(t)$ will be kept in the solution.

$$\ddot{\phi} + 2\delta\dot{\phi} + k_1\phi = k_b\zeta(t) + k_1\phi_{v0}\,. \quad (16)$$

As will be seen in the subsequent analysis, neglecting the influence of excitation no longer provides a significant simplification.

Expression (16) is an ordinary linear heterogeneous differential equation with a constant coefficient, allowing a close-form solution. Its total solution consists of the general solution of the homogeneous equation and a particular solution of the original heterogeneous equation.

$$\phi(t) = \phi_H(t) + \phi_G(t)\,. \quad (17)$$

The solution of the homogeneous equation is:

$$\phi_H(t) = A\exp(\lambda_1 t) + B\exp(\lambda_2 t)\,. \quad (18)$$

The constants A and B depend on initial conditions and will be defined later; the eigenvalues are

$$\lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - k_1} \quad (19)$$

The particular solution must be similar to excitation; *i.e.* a sum of the heave motions plus a constant:

$$\phi_G(t)=\sum_{i=1}^{N_\omega} p_{ai}\cos(\omega_i t+\gamma_{pi}+\varphi_i)+\phi_{v0} \\ = p(t)+\phi_{v0}\,. \tag{20}$$

The components of the particular solutions are defined in the usual way for a linear equation:

$$p_{ai}=\frac{\zeta_{ai}k_b}{\sqrt{(\omega_i^2-k_1)^2+4\delta^2\omega_i^2}} \\ \gamma_{pi}=\arctan\frac{2\delta\omega_0}{\omega_i^2-k_1}\,. \tag{21}$$

The arbitrary constants then can be expressed as:

$$A=\frac{(\dot\phi_1-\dot p_1)-\lambda_2\left(\phi_1-\phi_{v0}-p_1\right)}{\lambda_1-\lambda_2} \tag{22}$$

$$B=-\frac{(\dot\phi_1-\dot p_1)-\lambda_1\left(\phi_1-\phi_{v0}-p_1\right)}{\lambda_1-\lambda_2}\,. \tag{23}$$

Here ϕ_1 and $\dot\phi_1$ are initial conditions while p_1 and $\dot p_1$ are the values of the particular solution (20) and its derivative at the initial instant t_1. The initial instant is essentially the upcrossing instant, so:

$$\phi_1=\phi_m(t_1) \tag{24}$$

The complete general solution and its first derivative are

$$\phi(t)=A\exp(\lambda_1 t)+B\exp(\lambda_2 t)+p(t)+\phi_{v0} \tag{25}$$

$$\dot\phi(t)=\lambda_1 A\exp(\lambda_1 t)+\lambda_2 B\exp(\lambda_2 t)+\dot p(t) \tag{26}$$

4.2. Process of Critical Roll Rate

It is now necessary to introduce the concept of the critical roll rate, $\dot\phi_{cr}$, which is defined as the value of the initial roll rate at the up-crossing of the first threshold $\dot\phi_1$, which leads to crossing of the 2nd threshold and therefore capsizing. For every given instant of time and set of initial phases φ_i, there is a pair of initial conditions, $\phi_m(t);\dot\phi_{cr}$, that deterministically lead to having the roll angle reach the second threshold, ϕ_{m1}, beyond which the ship will capsize. It can be directly concluded from this definition that the critical roll rate is a stochastic process taking different values at different instant of time; it is also different for different realizations at the same instant of time.

The critical roll rate can be calculated numerically by an iterative algorithm, as described in (Belenky, et al., 2008).

Alternatively, the critical roll rate can be calculated from the condition:

$$A(\dot\phi_{cr})=0\,. \tag{27}$$

Equation (27) can be solved using formula (22); for any instant of time, t,

$$\dot\phi_{cr}(t)=\lambda_2\left(\phi_m(t)-\phi_{v0}-p(t)\right)+\dot p(t)\,. \tag{28}$$

Thus, the critical roll rate is a stochastic process defined at any instant of time. As formula (28) represents a linear combination of three normal processes, the distribution of roll rates is normal as well. The process of critical roll rate can then be presented in the form of a Fourier series. Technically, it is convenient to calculate the difference between the first two processes and then add the third one. Consider an auxiliary process $w(t)$ defined by the following formula:

$$w(t)=\lambda_2\left(\phi_m(t)-\phi_{v0}-p(t)\right). \tag{29}$$

This process is presented using a Fourier series as follows:

$$w(t)=\lambda_2(\phi_{m0}-\phi_{v0})+ \\ \sum_{i=1}^{N_\omega} w_{ai}\cos(\omega_i t+\gamma_{wi}+\varphi_i)\,. \tag{30}$$

The amplitudes of the components of the auxiliary process w are expressed as:

$$w_{ai} = \lambda_2 \Big(\phi_{mai}^2 + p_{ai}^2 - 2\phi_{mai}\phi_{1ai}\cos(\gamma_{\phi mi} - \gamma_{pi})\Big)^{0.5} \quad (31)$$

The phase shift for the auxiliary process, w, is expressed as:

$$\gamma_{zi} = \arctan\frac{\phi_{mai}\sin(\gamma_{\zeta i}) - \phi_{1ai}\sin(\gamma_{pi})}{\phi_{mai}\cos(\gamma_{\zeta i}) - \phi_{1ai}\cos(\gamma_{pi})} \quad (32)$$

The process of critical roll rates is now expressed as a simple sum of the auxiliary process, w, and the derivative of the particular solution, $\dot{p}_1$:

$$\dot{\phi}_{cr}(t) = w(t) + \dot{p}(t) \quad (33)$$

The process can be presented in the form of a Fourier *series*

$$\dot{\phi}_{cr}(t) = \lambda_2(\phi_{m0} - \phi_{v0}) + \sum_{i=1}^{N_\omega} \phi_{crai}\cos(\omega_i t + \gamma_{cri} + \varphi_i)\,. \quad (34)$$

The components of the Fourier presentation (34) are as *follows*:

$$\phi_{crai} = \Big(w_{ai}^2 + \omega_i^2 p_{ai}^2 - 2w_{ai}p_{ai}\omega_i\sin(\gamma_{wi} - \gamma_{pi})\Big)^{0.5} \quad (35)$$

$$\gamma_{cri} = \arctan\frac{w_{ai}\sin(\gamma_{wi}) - p_{ai}\omega_i\cos(\gamma_{pi})}{w_{ai}\cos(\gamma_{wi}) + p_{ai}\omega_i\sin(\gamma_{pi})}\,. \quad (36)$$

4.3. Process of Difference between Critical and Instantaneous Roll Rates

The condition of capsizing after upcrossing can be formulated as follows: when an upcrossing occurs, the roll rate at upcrossing should exceed the critical roll rate:

$$\dot{\phi}(t_1) > \dot{\phi}_{cr}(t_1)\,. \quad (37)$$

It makes sense, therefore, to consider a process of the difference between the instantaneous and critical roll rates

$$\dot{\phi}_d(t) = \dot{\phi}_{cr}(t) - \dot{\phi}(t) \quad (38)$$

The process of the difference, further identified as $\dot{\phi}_d(t)$, is normal as both critical and instantaneous roll rates are normal.

An assumption that upcrossing follows Poisson flow infers that upcrossings are rare; it allows the use of a particular solution of the non-rare problem, because the general solution of homogeneous equation generated after each down-crossing will not be statistically significant. Therefore,

$$\dot{\phi}(t) = \sum_{i=1}^{N_\omega} \phi_{ai}\omega_i\cos(\omega_i t + \gamma_{\phi i} + \varphi_i)\,. \quad (39)$$

The process of the difference of rates can now be trivially presented as a Fourier series as it is already defined for both processes on the right-hand-side of equation (38):

$$\dot{\phi}_d(t) = \lambda_2(\phi_{m0} - \phi_{v0}) + \sum_{i=1}^{N_\omega} a_{rdi}\cos(\omega_i t + \gamma_{rdi} + \varphi_i)\,. \quad (40)$$

The components of this formulation are defined as follows:

$$a_{rdi} = \Big(\phi_{crai}^2 + \omega_i^2\phi_{ai}^2 - 2\phi_{crai}\phi_{ai}\omega_i\cos(\gamma_{cri} - \gamma_{\phi i})\Big)^{0.5} \quad (41)$$

$$\gamma_{rdi} = \arctan\frac{\phi_{crai}\sin(\gamma_{cri}) - \phi_{ai}\omega_i\sin(\gamma_{\phi i})}{\phi_{crai}\cos(\gamma_{cri}) - \phi_{ai}\omega_i\cos(\gamma_{\phi i})}\,. \quad (42)$$

4.4. Probability of Capsizing

Following the formulation of the condition of capsizing after upcrossing (37), its probability can be expressed as:

$$P_{C|U} = \int_{-\infty}^{0} f_{cr}\left(\dot{\phi}_d\right) d\dot{\phi}_d \ . \qquad (43)$$

Here $f_{cr}\left(\dot{\phi}_d\right)$ is the distribution of the distance between critical and instantaneous roll rates at the instance of upcrossing. This distribution can be expressed as (see derivation in Appendix 1):

$$f_{cr}(\dot{\phi}_d) = \frac{\int_0^{\infty} \dot{x} f(x = \phi_{m0}, \dot{x}, \dot{\phi}_d) d\dot{x}}{f(x = \phi_{m0}) \int_0^{\infty} \dot{x} f(\dot{x}) d\dot{x}} \ . \qquad (44)$$

Here $f(x)$ and $f(\dot{x})$ are the probability density distributions of the carrier and its derivative. These distributions are known to be normal. The term, $f(x, \dot{x}, \dot{\phi}_d)$, expresses a joint distribution of the carrier process, its derivative and the distance between instantaneous and critical roll rates. As marginal distributions of all these processes are normal, it is logical to assume that their joint distribution is also normal. Then their mutual dependence can be fully characterized by correlation moments. Evaluation of these correlation moments does not present any difficulties as Fourier presentations are available for all of them:

$$M\left(x, \dot{\phi}_d\right) = 0.5 \sum_{i=1}^{N_\omega} x_{ai} a_{rdi} \cos(\gamma_{xi} - \gamma_{rdi}) \ , \qquad (45)$$

$$M\left(\dot{x}, \dot{\phi}_d\right) = 0.5 \sum_{i=1}^{N_\omega} x_{ai} a_{rdi} \omega_i \sin(\gamma_{xi} - \gamma_{rdi}) \ , \qquad (46)$$

and

$$M\left(x, \dot{x}\right) = 0 \ . \qquad (47)$$

Respectively, correlation coefficients are:

$$r_{xd} = \frac{M\left(x, \dot{\phi}_d\right)}{\sigma_x \sigma_d}; \quad r_{\dot{x}d} = \frac{M\left(\dot{x}, \dot{\phi}_d\right)}{\sigma_{\dot{x}} \sigma_d} . \qquad (48)$$

Standard deviations are:

$$\sigma_x = \sqrt{V_x}; \quad \sigma_{\dot{x}} = \sqrt{V_{\dot{x}};} \quad \sigma_d = \sqrt{V_d} \ . \qquad (49)$$

The variances of the carrier process and its derivative are defined by formulae (14). The variance of the distance between instantaneous and critical roll rates can be found from its Fourier presentation (40):

$$V_d = 0.5 \sum_{i=1}^{N_\omega} a_{rdi}^2 \ . \qquad (50)$$

As a formula for tri-variate normal distribution is rather cumbersome, it is more convenient to consider it as a product of marginal bi-variate and conditional distributions:

$$f(x, \dot{x}, \dot{\phi}_d) = f(\dot{x} \,|\, x, \dot{\phi}_d) f(\dot{\phi}_d, x) \ . \qquad (51)$$

The first term in (51) is a conditional distribution of the derivative of the carrier if two other processes have taken particular values. As the tri-variate distribution is normal, the conditional distribution is normal too. Its parameters are defined through the parameters of the tri-variate distribution in the following way:

$$m_{\dot{x}|xd} = \frac{\sigma_{\dot{x}}}{1 - r_{xd}^2} \left(\frac{(\dot{\phi}_d - m_d) r_{\dot{x}d}}{\sigma_d} - \frac{\phi_{m0} r_{\dot{x}d} r_{xd}}{\sigma_x} \right) \qquad (52)$$

and

$$\sigma_{\dot{x}|xd} = \sigma_{\dot{x}} \frac{\sqrt{1 - r_{xd}^2 - r_{\dot{x}d}^2}}{\sqrt{1 - r_{xd}^2}} \ . \qquad (53)$$

Here m_d is a mean value of the distance between instantaneous and critical roll rate. It is readily available from (40):

$$m_d = \lambda_2 (\phi_{m0} - \phi_{v0}) \ . \qquad (54)$$

The standard deviation (53) of the conditional distribution is a constant, while the mean value is a function of $\dot{\phi}_d$; therefore, as expected, the conditional distribution in (51) is a function of two variables:

$$f(\dot{x} \mid x = \phi_{m0}, \dot{\phi}_d) = f_{m0}(\dot{x} \mid \dot{\phi}_d) =$$
$$\frac{1}{\sigma_{\dot{x}|xd}\sqrt{2\pi}} \exp\left(-\frac{\left(\dot{x} - m_{\dot{x}|xd}(\dot{\phi}_d)\right)^2}{2\sigma^2_{\dot{x}|xd}}\right). \quad (55)$$

The second term in (51) is just a bi-variate normal distribution of the carrier and the distance between the instantaneous and critical roll rates, evaluated at the angle of the maximum of the restoring curve in calm water, $x = \phi_{m0}$. It is convenient to present it in the form of a conditional distribution:

$$f(\dot{\phi}_d \mid x = \phi_{m0}) = \frac{f(\dot{\phi}_d, x = \phi_{m0})}{f(x = \phi_{m0})}. \quad (56)$$

Distribution (56) is also normal with mean value and standard deviation defined as:

$$m_{d|x} = \phi_{m0} \frac{\sigma_d}{\sigma_x} r_{xd} \quad (57)$$

and

$$\sigma_{d|x} = \sigma_d \sqrt{1 - r_{xd}^2}. \quad (58)$$

The conditional distribution (56) can then be expressed as follows:

$$f(\dot{\phi}_d \mid x = \phi_{m0}) = f_{m0}(\dot{\phi}_d) =$$
$$\frac{1}{\sigma_{d|x}\sqrt{2\pi}} \exp\left(-\frac{\left(\dot{\phi}_d - m_{d|x}\right)^2}{2\sigma^2_{d|x}}\right). \quad (59)$$

As the distribution of the derivative of the carrier is also normal, the integral in the denominator in (44) can be evaluated in closed form:

$$\int_0^\infty \dot{x} f(\dot{x}) d\dot{x} = \frac{\sigma_{\dot{x}}}{\sqrt{2\pi}}. \quad (60)$$

The substitution of formulae (51), (55), (59) and (60) into (44) gives the final formula for the distribution of the distance between instantaneous and critical roll rate at the upcrossing of the carrier:

$$f_{cr}(\dot{\phi}_d) = \frac{\sqrt{2\pi}}{\sigma_{\dot{x}}} f_{m0}(\dot{\phi}_d) \int_0^\infty \dot{x} f_{m0}(\dot{x} \mid \dot{\phi}_d) d\dot{x}. \quad (61)$$

Finally, the probability of capsizing during time T is expressed by combinating (12) and (43),

$$P_C(T) = 1 - \exp(-\xi P_{C|U} T). \quad (62)$$

5. SELF-CONSISTENCY CHECK

In order to ensure that the theoretical solution is correct, a self-consistency check was performed. In this check, the dynamical system described by (4), with stiffness defined by (2) and (3), was evaluated for a large number of records in the time domain and the number of observed capsizes was counted in order to get a statistical probability of capsize. The results of the study, which are presented below, demonstrate clear convergence of the statistical probability of capsize to the theoretical solution (62). In addition to the final probabilities, some intermediate results are also presented in order to demonstrate how the theory works.

Numerical data for the self-consistency check, including ship and wave properties, are shown in Table 1. The irregular seaway was derived using a Bretschneider open ocean wave spectrum. Figure 6 shows the autocorrelation function of waves evaluated from the spectrum using cosine Fourier transform on the accepted discretization. It shows no sign of a self-repeating effect and therefore demonstrates the statistical representativeness of the accepted spectrum discretization for the desired duration of the time records (30 min). The initial set of calculations consisted of 200 records, each representing a 30 minutes realization of the same irregular seaway, and was used to check intermediate results such as the characteristics of upcrossing as well as the final capsizing result. Two sets of 6000 records each were then used to check the convergence of the statistical probability of capsizing.

Table 1. Numerical data for self-consistency test.

Variable	Symbol	Unit	Value
Roll natural frequency,	ω_0	1/s	0.65
Heave natural frequency	ω_ζ	1/s	1.3
Angle of maximum in calm water	ϕ_{m0}	Rad	0.5
Roll damping (fraction of critical)	μ_ϕ	-	0.2
Heave damping (fraction of critical)	μ_ζ	-	0.4
Slope of decreased part of stiffness	k_1	-	1.0
Angle of vanishing stability in calm water	ϕ_{v0}	Rad	1.0
Slope coefficient for stability change	k_d	1/m	0.0256
Duration of each record	T	s	1800
Time step	Δt	s	0.2
Number of frequencies	N_ω	-	245
Significant wave height	H_s	m	11.5
Modal wave period	T_m	s	16.4

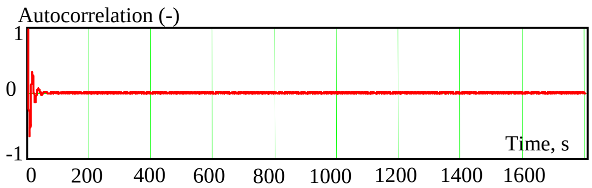

Figure 6. Autocorrelation function of waves calculated using Fourier cosine transform from input spectrum.

Figure 7 shows one of the capsizing episodes observed in this calculation set. The upper graph shows the time histories of the ship roll angle (ϕ) and the instantaneous angle of the maximum of the restoring curve (ϕ_m). At approximately $t = 1425$ s, the roll angle exceeds the instantaneous angle of maximum of the restoring curve. The lower graph shows the time histories of the critical, $(\dot{\phi}_{cr})$, and instantaneous roll, $(\dot{\phi})$, rates. At that time, the instantaneous roll rate exceeded the critical roll rate. As a result, the dynamical system

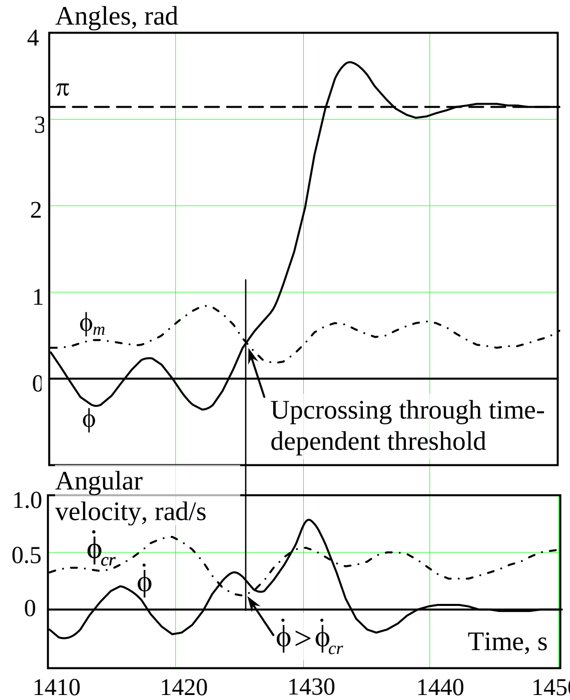

Figure 7. Capsizing episode: the roll crosses instantaneous angle of maximum, and instantaneous roll rate exceeded critical roll rate.

experiences capsizing and transits to another equilibrium state.

Applicability of the Poisson flow to the upcrossing events of the carrier process is one of the key assumptions in the method. This can be checked by comparing the cumulative distribution of the time before upcrossing with the theoretical distribution and using the Kholmogorov-Smirnov (K-S) test to judge the goodness-of-fit. The test gives a 95% probability that the difference between the two is a result of random causes, which confirms the fit. Figure 8 shows the theoretical curve calculated with equation (12) along with statistical points, representing a probability that at least one upcrossing has been observed during that time. The inset in Figure 8 compares the theoretical rate of upcrossing (13) to the mean number of observed upcrossings per unit of time shown with a 95% confidence interval. Justification and more details on these procedures can be found in (Belenky, *et al.*, 2008).

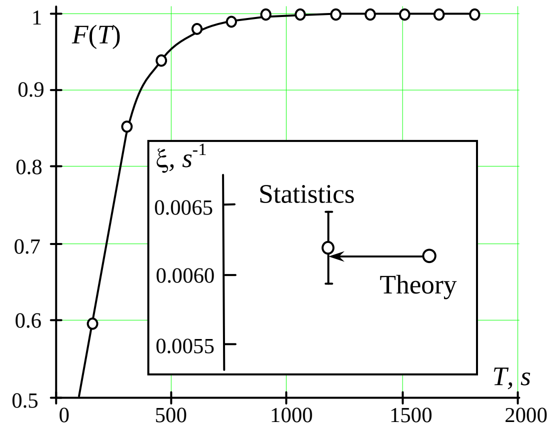

Figure 8. Cumulative distribution of the time with at least one upcrossing. Probability that the fit is good is 0.95 (K-S Test); inset: upcrossing rate.

Another important point to check is the distribution of the difference between instantaneous and critical roll rate at upcrossing of the carrier (61) as well as probability of capsizing after upcrossing (43). Figure 9 shows a histogram of the values of the difference between the instantaneous and critical roll rate taken at the instant of upcrossing. The theoretical curve is calculated with formula (61). The importance of the difference between theoretical results and observed statistics was judged using Pierson chi-square criterion that has shown a finite probability of 7.7% that the observed difference is caused by random

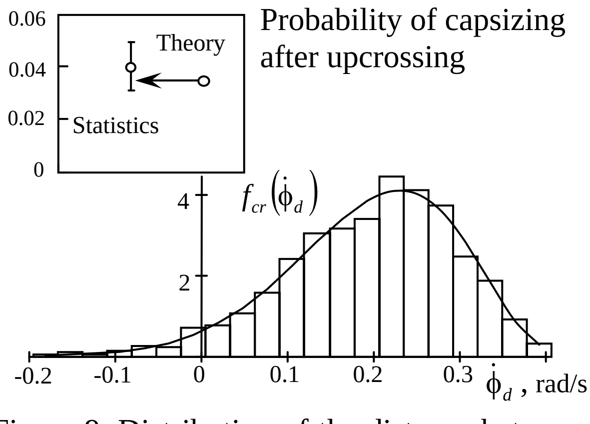

Figure 9. Distribution of the distance between critical and instantaneous roll rate at upcrossing. Probability that the fit is good is 0.077.

reasons.

The inset in Figure 9 shows comparison of the theoretical probability of capsizing after upcrossing (43) with the statistical estimate (Out of 200 runs, 70 ended up with capsizing and there were 1783 upcrossings all in total).

The boundaries of the confidence interval were evaluated using the standard formula for statistical frequency:

$$P_{u,l} = \frac{P^* + \dfrac{K_\beta^2}{2N} \pm K_\beta \sqrt{\dfrac{P^*\left(1-P^*\right)}{N} + \dfrac{K_\beta^2}{4N^2}}}{1 + \dfrac{K_\beta^2}{N}} \quad (63)$$

Here P^* is the statistical frequency, N is the volume of the sample, and K_β is the half-breadth is expressed in terms of standard deviations; for the accepted confidence probability of 95%, $K_\beta = 1.95996$.

Figure 10 shows a comparison between the theoretical and statistical estimates of probability of capsizing during a period of 30 minutes. As can be seen from this figure, the theoretical probability of 0.32 is within of the confidence interval of the statistical frequency of capsizing 0.35 ± 0.07 estimated over 200 independent realizations.

To complete the self-consistency check, the convergence of the probability was tested. As shown in Figure 11, the statistical probability of capsizing was evaluated for an increasing number of records and plotted along with their confidence interval and the theoretical solution (63). The upper and lower graphs correspond to two independent sets of initial phases for the waves. These two sets were used to see different patterns of convergence.

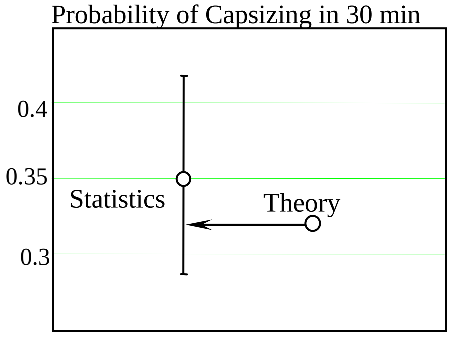

Figure 10. Probability of Capsizing in 30 min estimated from statistics over 200 records and calculated with formula (62).

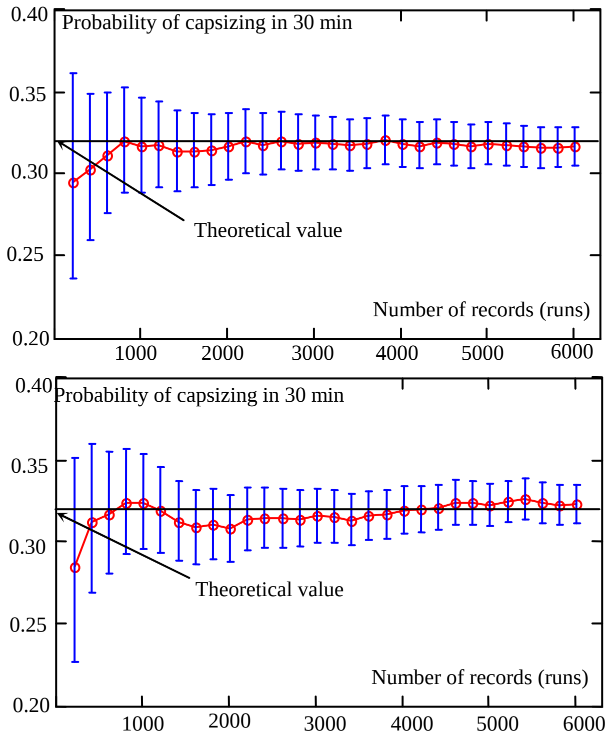

Figure 11. Convergence of the statistical frequency to theoretical solution as number of runs increases. Upper and lower graphs differ in sets of initial phases.

As it can be seen from both graphs in Figure 11, the statistical probability of capsizing does converge and the theoretical result stays within the confidence interval of the statistical value. The convergence is relatively fast in the first 1000 records but slows down afterwards. This may be partially caused by imperfect phase generation as capsizing phenomenon is known to be very sensitive to phases.

Taking into account both the analysis of intermediate results such as upcrossing rate and a verification of the convergence of the statistical probability of capsizing to the theoretical value, the self-consistency check for the present method may be considered as an affirmation in the sense that it does not disapprove the theory.

6. CONCLUSIONS

This paper presents the application of a split-time approach to evaluating the probability of a ship capsizing in waves with consideration of the ship’s change in stability in waves.

The change in stability for a ship in waves can be modeled in a dynamical system with piecewise linear stiffness by considering the boundary between ranges as a stochastic process correlated with the excitation. This process may be introduced as a deterministic function of heave motions. If the increasing range remains linear and the decreasing part remains parallel to itself all the time, the solution remains linear within each range, and an analytic solution to the roll equation can be presented.

The concept of critical roll rate as a stochastic process has been introduced. The critical roll rate is defined as the roll rate at which a ship upcrossing the maximum of the GZ curve will capsize. Since the GZ curve is changing with the motion in waves, the critical roll rate is a function of time. At each instant of time, if an upcrossing occurs and the roll rate at upcrossing exceeds the critical roll rate, then a capsizing is imminent.

The capsizing probability can therefore be associated with the probability of upcrossing the maximum of the GZ curve with a roll rate at the instant of upcrossing exceeding the critical roll rate. Upcrossings are assumed to follow Poisson flow.

The self-consistency of the method was demonstrated by evaluating the response of the dynamical system in time domain, counting the observed capsizes to get a direct statistical estimate of the probability of capsizing, and demonstrating that this value converges to the theoretical result as the volume of statistical data is increased.

In general, the viability of a split-time approach considering the change of stability in waves was demonstrated using a simple example. This paves the way to the application of the split-time approach with more sophisticated tools including advanced numerical simulations of ships in waves.

7. ACKNOWLEDGEMENTS

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9. APPENDIX 1

Consider a stationary stochastic process, $x(t)$, that crosses a level a at an arbitrary instant of time t. Consider another stochastic process, $y(t)$, that depends on the process $x(t)$. The objective is to find the probability density distribution of the instantaneous value of the process $y(t)$ when the process $x(t)$ up-crosses the level a.

A random event of upcrossing is defined as:

$$U = \begin{cases} x(t) < a \\ x(t+dt) > a\,. \end{cases} \quad \text{(A1)}$$

By definition, the cumulative probability distribution is:

$$F_{cr}(y) = P(y \le b \,|\, U)\,. \quad \text{(A2)}$$

The conditional probability in formula (A2) can be expressed as:

$$P(y \le b \,|\, U) = \frac{P(y \le b \cap U)}{P(U)}\,. \quad \text{(A3)}$$

Here $P(y \le b \cap U)$ is the probability of occurrence of an upcrossing with the value of the process, $y(t)$, not exceeding an arbitrary number b. This random event can expressed through the following system of inequalities:

$$y \le b \cap U = \begin{cases} x(t) < a \\ x(t+dt) > a \\ y \le b \end{cases} = \begin{cases} x(t) < a \\ x(t) > a - \dot{x}dt \\ y \le b\,. \end{cases} \quad \text{(A4)}$$

Probability of the random event defined by equation (A4) can be expressed trivially through a joint distribution of the process $x(t)$, its *derivative* and the process $y(t)$:

$$P(y \le b \cap U) = \int_{-\infty}^{b}\int_{0}^{\infty}\int_{a-\dot{x}dt}^{a} f(x,\dot{x},y)\,dx\,d\dot{x}\,dy\,. \quad \text{(A5)}$$

The most internal integral in the formula (A5) has limits *that* are infinitely close to each other. Application of the Integral Mean Value Theorem yields:

$$P(y \le b \cap U) = dt\int_{-\infty}^{b}\int_{0}^{\infty} \dot{x} f(a,\dot{x},y)\,d\dot{x}\,dy\,. \quad \text{(A6)}$$

The probability of upcrossing $P(U)$ can be expressed in similar way:

$$P(U) = dt\, f(a)\int_{0}^{\infty} \dot{x} f(\dot{x})\,d\dot{x}\,. \quad \text{(A7)}$$

The cumulative distribution of the value of y(t) at upcrossing can be expressed by substituting (A6) and (A7) into (A3) and (A2):

$$F_{cr}(y) = \frac{\int_{-\infty}^{b}\int_{0}^{\infty} \dot{x} f(a,\dot{x},y)\,d\dot{x}\,dy}{f(a)\int_{0}^{\infty} \dot{x} f(\dot{x})\,d\dot{x}}\,. \quad \text{(A8)}$$

The probability density is obtained from (A8) by taking a derivative with respect to y:

$$f_{cr}(y) = \frac{\int_{0}^{\infty} \dot{x} f(a,\dot{x},y)\,d\dot{x}}{f(a)\int_{0}^{\infty} \dot{x} f(\dot{x})\,d\dot{x}}\,. \quad \text{(A9)}$$

Equation (A9) represents the final result for the distribution of the dependent process at upcrossing. This result may be known in the Upcrossing Theory; however the authors were unable to locate appropriate references.