Development of Reduced Order Models for Hydrodynamic Responses

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ABSTRACT

Hydrodynamic prediction codes based on potential flow or RANS have matured to a level that they can readily be applied to many engineering level analyses, but are still too expensive to directly apply to many extreme response problems. One potential solution is to implement a multi-fidelity framework which uses higher fidelity models to develop Reduced Order Models (ROMs) of different types and then use those ROMs to develop extreme response models and identify conditions leading to extreme response events. This paper presents several ideas about the characteristics of effective ROMs and quantifying the uncertainty of ROMs in the multi-fidelity approach.

Keywords: *Reduced order model, Uncertainty quantification.*

1. INTRODUCTION

A principal feature of any reduced-order model (ROM) is that it represents a reduction or "step back" in computational complexity. This may seem counterintuitive – since computer modelling was first introduced into Naval Architecture practice, progress in the prediction of dynamic stability, motions, and structural loads has almost universally been associated with an increase in the complexity of the mathematical models; see a review of Beck and Reed (2001) as well as Reed and Beck (2016).

The development of computational methods for the prediction of ship motions and loads in irregular waves has been a focus of the Naval Architecture community since the publication of St. Denis and Pierson (1953). Frequency domain methods including diffraction and radiation forces became available in the early 1970's (*e.g.* Salvesen *et al.*, 1970). Full consideration of nonlinearity of hydrostatic and Froude-Krylov forces leads to a transition from the frequency domain to the time domain. Computational methods based on potential flow hydrodynamics were developed (*e.g.* de Kat and Paulling, 1989; Lin and Yue, 1990). These methods have enabled hybrid codes, combining the body-nonlinear formulation for hydrostatic and Froude-Krylov forces with boundary-value solutions for diffraction and radiation either in body-linear or nonlinear formulation (*e.g.* Shin *et al.*, 2003; Belknap and Reed, 2019).

The most complete numerical solution of the hydrodynamic body-nonlinear formulation available today involves solving the Navier-Stokes equation for the flow around the hull, usually with averaging of the Reynolds stresses (RANS), with a nonlinear free surface boundary condition at the air water interface. Advanced RANS codes are capable of providing a very high fidelity solution for ship motions (*e.g.* Gorski *et al.*, 2014; Aram and Kim, 2017). The computational cost of RANS, however, makes its application for irregular wave ship motion assessment impractical. At the same time, RANS provides a practical source of data for building models of viscous and vortical forces (*e.g.* roll damping and maneuvering forces) for potential flow codes and stand-alone dynamic solvers (*e.g.* Hughes *et al.*, 2019; Aram and Silva, 2019; Aram and Wundrow, 2022).

In the latter case, the RANS calculations are used in lieu of a model test. Generally, this is nothing new: experimental data have long been used to present forces that were not directly available through computation. Now these forces are being pre-computed. Essentially, two models of different fidelity (potential flow codes and RANS) are being used together in a "hybrid" manner.

Extreme ship response, defined as the largest motions, accelerations, or loads that might be encountered in a particular set of conditions, are of special interest for both designers and operators. With the development of time domain solvers, direct Monte-Carlo approaches seem to be the most evident way to obtain information on extreme response. However, the computational cost of direct Monte-Carlo approaches is still too high even for the case of hybrid flow solvers. For example, the Large Amplitude Motion Program (LAMP – Shin *et al.*, 2003) runs on the order of real time, so a reliable quantification of an extreme response may require thousands of hours of simulation data and, therefore, thousands of hours of computational time. Thus, in order to get to extremes, one must either use statistical extrapolation or further simplify the mathematical model to improve computational speed. The latter option seems to be unreasonable, because the extreme event is likely to be when an accurate evaluation is most needed.

Can ROMs be used as a predictor of extreme events? Reed (2021) demonstrated that they can, when used in conjunction with higher fidelity tools: a volume-based method was run to identify wave records where extreme events are likely. LAMP was then used to compute the actual response. In some sense, such a "ROM-as-predictor" method is akin to the wave group approach by Themelis and Spyrou (2007) and sequential sampling by Mohammad and Sapsis (2018).

In general, the multi-fidelity approach can be seen as a systematic framework of using models of different fidelity to their best efficiency; see Figure 1.

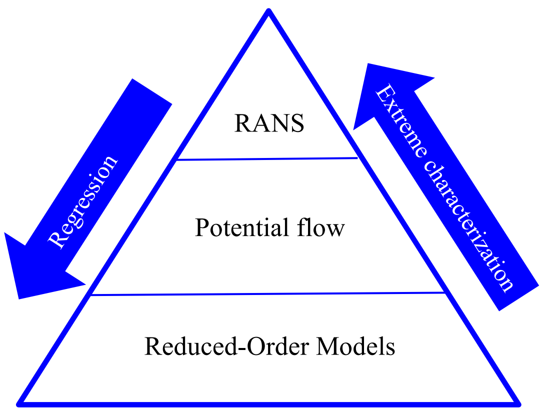

Figure 1: Framework of multi-fidelity extreme characterization

The framework shown in Figure 1 is already in use, as regression is used to extract data from RANS for roll damping (*e.g.* Aram and Park, 2022) and for maneuvering derivatives (*e.g.* Aram and Silva, 2019). LAMP was used in Pipiras *et al.* (2022) to regress diffraction and radiation, while the volume-based SimpleCode is employed to characterize extremes (Reed, 2021)

To be practical, the multi-fidelity framework requires consistency between the models of different fidelity – the models much solve the same problem and produce results that are complementary from level to level. At the same time, different sets of assumptions in models of different fidelities leave very little chance for exactly the same result. Each level of simplification brings modeling uncertainty. While the consistency of the models can be generally established through validation exercises, the consistency of specific assessments can be fully defined only if the uncertainty of ROMs has been quantified.

2. REVIEW OF ROMS

This section reviews the general ideas behind the development of "successful" ROMs to determine if any general principles can be distilled. "Successful" ROMs are understood to be the models or methods in which simplifications lead to new functionality or new knowledge. As the objective is to understand the underlying principle, the review goes slightly outside of the stability field to also cover seakeeping, maneuvering, and wave loads.

Two types of ROMs can be identified in the literature: semi-analytical and numerical. The distinction is somewhat academic, as the final result is produced by numerical method anyway.

Semi-Analytical ROMs

Semi-analytic ROMs are highly reduced models which are simple enough to allow an analytic or nearly analytic solution, which can provide a direct evaluation of the probability of an extreme event or the distribution of extreme responses. While generally too simple to provide a quantitatively accurate result, such ROMs can be essential tools in the development procedures and tools for use with higher fidelity tools.

An example of this is the development of the split-time method for the probability of capsizing in irregular waves (Weems *et al*., 2022). Estimating the probability of capsizing is a very difficult problem from the numerical point of view: it combines an extreme rarity of event and a very large degree of nonlinearity. The essential idea of the split-time method, which involves splitting the problem into non-rare and rare parts, was derived from a ROM with a piecewise linear approximation of the roll restoring (GZ) curve (Belenky, 1993). While simple, the ROM with piecewise linear GZ curve is capable of modelling the key feature of the problem, which is two stable equilibria and the transition between them; see Figure 2 (Belenky *et al.*, 2016). Weems *et al*. (2022) shows that the split-time method was successful for a limited statistical validation (Smith, 2019).

The piecewise linear ROM also helped to determine the tail structure of distribution of large roll angles (Belenky *et al.,* 2019). Knowledge that large roll angles (*i.e.* in vicinity of maximum of the GZ curve) are likely to have a distribution with a heavy tail allows the construction of a physics-informed scheme for extrapolation using envelope peak over threshold (EPOT), which has shown reasonable results in stern-quartering and following seas (Campbell *et al.,* 2022). A general principle used in piecewise linear ROM is schematization – constructing the simplest possible model that reproduces the essential physics of roll motion and capsizing, which in this case is the existence of two stable equilibria.

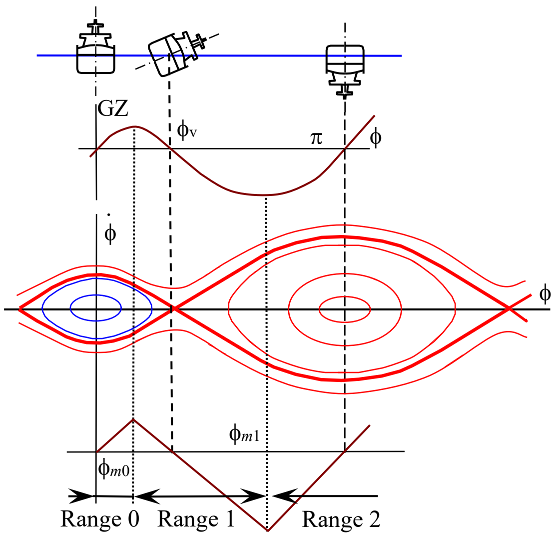

Figure 2: Phase plane topology of capsize and piecewise linear stiffness (Belenky *et al.*, 2016)

The existence of the equilibria defines the topology of the phase plane, and is "responsible" for the most basics physics of the phenomenon. That is why a single degree of freedom dynamical system describing surging and surf-riding was sufficient for Spyrou (1996) to relate the surf-riding phenomenon to homoclinic bifurcation (see also Spyrou, 2017).

A focus on the topology of the phase plane allowed Maki (2017) to obtain the shape of roll motion distribution. He showed that while it is critical that the ROM include the principle non-linearity associated with restoring, the bandwidth of the excitation was not that important for the distribution shape, including its tails. In fact, the presentation of the excitation as white noise can be considered as a schematization of excitation.

In Sapsis *et al.* (2020), simultaneous hydrostatic and excitation schematizations were applied to develop a ROM for the hydrostatic and incident wave (Froude-Krylov) heave force and pitch moment in longitudinal waves. The idea is to represent the station lines by a second-order Taylor series and approximate irregular seas with a two-component wave with the same frequency and white-noise amplitudes. The frequency is set to have a wave length equal to the ship length. This ROM led to semi-analytical formula for probability density function (PDF) for wave-induced vertical bending moment and demonstrated that the asymmetry of PDF of VBM is driven by the angle of a station on a waterline. Sapsis *et al.* (2022) and Belenky *et al.* (2022) further extended this ROM to account for the effect of deck submergence.

A classic example of schematization of excitation is Grim's effective wave (Grim, 1961), where a longitudinal profile of irregular seas is approximated with a single wave with a length equal to the ship length and with random amplitude. The amplitude is set to achieve an equivalent variation of stability. Umeda and Yamakoshi (1986, 1994) have demonstrated the accuracy of Grim's effective wave, and proposed the inclusion of a surging effect into the calculations in order to account for the timing of the exposure to reduced stability conditions. Further improvements to Grim's effective wave are described in Bulian (2008).

Schematization of excitation is not limited to waves. Sapsis *et al.* (2021) uses a delta-function to model a slamming impact. Coupled with a Gaussian assumption of heave and pitch motion to determine slamming events and an elastic beam model of the ship structural response, a PDF of the impact-induced VBM can be obtained.

Schematization of excitation becomes especially effective when waves irregularity becomes essential and even changes the physics of the phenomenon. Parametric roll resonance is exactly such a phenomenon. The simplest model of parametric resonance is the Mathieu equation, which is a linear ordinary differential equation with a periodic coefficient describing the parametric excitation. When the frequency of the parametric excitation corresponds to an “instability” interval, the solution has no limit. In order to observe a finite steady state amplitude, a restoring nonlinearity must be present – detuning takes the system out of instability conditions.

The situation is quite different in irregular waves. The detuning can be modeled simply by wave randomness. That is the main idea of the intermittent instabilities approach developed by Mohamad and Sapsis (2016). The result recovered the characteristic shape of the PDF of parametric roll, which has been observed by Hashimoto *et al.* (2011) in a model test and by Belenky *et al.* (2011) in numerical simulations.

Another case of a substantial change of physics introduced by irregular waves is exhibited by surf-riding. Surf-riding is essentially a dynamic equilibrium created by the balance between thrust, resistance at wave celerity, and the Froude-Krylov surging force. Modeling the surging Froude-Krylov force in irregular waves (*e.g.* Belenky *et al.,* 2019a) and the celerity of irregular waves (Spyrou *et al.,* 2019) is not trivial – several options have been considered in the cited references including tracking maximum wave slope and definition through instantaneous frequency. The most important feature of surf-riding in irregular waves is that the point where the sum of the surging forces equals zero is no longer an equilibrium. Due to the stochastic character for Froude-Krylov forces and wave celerity, this point appears, disappears, and changes location in the phase plane, *i.e.* moves with acceleration. Thus the frame of references fixed to this point is no longer inertial. Consideration of bi-chromatic waves reveals very complex dynamics (Spyrou *et al.,* 2016).

Numerical ROMs

Weems and Wundrow (2013) and Weems and Belenky (2018) describe a volume-based approach to efficiently model nonlinear hydrostatic and Froude-Krylov forces in the time domain. This body-nonlinear formulation led to a very fast ship motion code commonly referred to as SimpleCode. It can serve as an example of a numerical ROM. Diffraction and radiation forces are approximated with polynomials, with coefficients determined by regressing LAMP-generated data (Pipiras *et al.,* 2022). Vortical forces are approximated by regressing RANS data (Silva and Aram, 2018; Aram and Silva, 2019; Weems *et al.,* 2020). Levine *et al.* (2022) and Howard *et al.* (2022) have demonstrated that a neural network can be efficiently used to post-correct the SimpleCode results, bringing it closer to an engineering-level potential flow code, in this case LAMP.

The volume-based body-nonlinear formulation is the only substantial difference between the SimpleCode and ordinary differential equation (ODE) models of ship motions. The ODE approach uses linear ship motion equations where nonlinear calm-water restoring is artificially introduced (*e.g.* Belenky and Sevastianov, 2007). As a result, hydrostatic and Froude-Krylov are artificially separated in the pure ODE models and it becomes difficult to model stability variation in irregular waves – Grim’s effective wave becomes the only realistic option. The volume-based approach allows the stability variation in irregular waves to be modeled without any additional assumptions.

Weems and Wundrow (2013) estimated the computational speed of the SimpleCode as 10 full-scale hours for 7 seconds on a single CPU core. There was no specific benchmarking of the SimpleCode against ODE-based simulation, but any gain in computational speed for the ODE model is most probably not worth the simplification in hydrostatic and Froude-Krylov forces, which are believed to be the most important nonlinearity in ship dynamics in waves.

The volume-based approach in the SimpleCode is essentially a transition from pressure to volume integration; the pressure decay in wave (Smith effect) is lost during such transition. One can characterize this transition as some sort of schematization of hydrostatic and Froude-Krylov forces.

Regression of other hydrodynamic forces also can be seen as schematization. Spyrou *et al.* (2009) mentions the body-nonlinear formulation for hydrostatic and Froude-Krylov forces based on pressure integration, while all other hydrodynamic forces are approximated with polynomials. This formulation was implemented in LAMP as LAMP-0. In terms of computational speed, LAMP-0 holds an intermediate position between the SimpleCode and the full version of LAMP where diffraction and radiation forces are found through the potential flow solution of the boundary-value problem.

Neither LAMP-0 nor SimpleCode model hydrodynamic memory. Spyrou *et al.* (2009) describe applying LAMP-0 in a 6 degree of freedom formulation with the continuation method in order to study surf-riding in stern quartering regular waves. Spyrou and Tigkas (2011) and Tigkas and Spyrou (2011) further extended continuation to include hydrodynamic memory effects. This can be done by introducing 40 additional degrees of freedom, *i.e.* by increasing the dimensionality of the problem. This demonstrates how the development of a simpler, no-memory ROM can be considered to be a reduction in the dimensionality of the problem.

Following this principle, the critical wave group approach (Themelis and Spyrou, 2007) can be seen as a numerical ROM developed by decreasing the dimensionality of a stochastic process in irregular waves. The latter is fully characterized by a joint distribution of all time sections, while a wave group can be defined by a limited number of random parameters such as number of waves, height and period of the largest wave in a group, etc. The probabilistic relationship of waves within a group is modeled with the Markov process, which can also be seen as a reduction of dimensionality (*e.g.* Anastopoulos and Spyrou, 2019).

Application of an auto regression / moving average (ARMA) method to model the wave field can also be seen as a reduction of dimensionality. Memory in space and time may be limited to 7 to 9 instances; see *e.g.* Weems *et al.* (2016) and Degtyarev *et al.* (2019).

Reducing dimensionality in the form of a wave group presentation allows Cousin and Sapsis (2016) to find a ROM-precursor of rogue waves by considering the interaction between modulation instability properties of localized wave groups and the statistical properties of the wave groups. Farazmand and Sapsis (2017) extended this approach to short-crested seas.

Directly reducing dimensionality through non-parametric Gaussian Process Regression (GPR) was used by Wan *et al.* (2018) to develop a data-assisted ROM approach for extreme events in a complex dynamical system. Dimensionality is reduced by a projection of the high-dimensional parameter space into low-dimensional parameter space; a review is also available from Sapsis (2018).

Similar principles are behind sequential sampling, developed by Mohamad and Sapsis (2018), where GPR is used to find a sequence of waves that is likely to invoke an extreme event. Silva and Maki (2022) use a neural network, trained with LAMP results, as a surrogate for ship dynamic response to determine critical wave groups.

General Prinicples of ROMs

The review of ROMs in the previous two subsections is far from complete. Nevertheless, it helps to distill several ideas that have led to successful ROMs:

- Schematization of forces, thereby preserving topology of phase plane, hydrostatic, and Froude-Krylov forces, in the most cases
- Schematization of excitation, especially when irregular waves substantially change the phenomenon
- Reducing dimensionality of the space of parameters.

Some of the reviewed ROMs can be envisioned as part of a workflow for assessing extreme events, shown in Figure 3.

3. APPROACH TO QUANTIFICATION OF MODELING UNCERTAINTY

The efficiency and effectiveness of the multi-fidelity framework, shown in Figure 1, comes at a price. This price is a requirement for a certain level of consistency and accuracy in the different levels of modeling. In the extreme event assessment framework, it means that the largest response conditions predicted by the lower fidelity model remain the largest response conditions when predicted by the higher fidelity model.

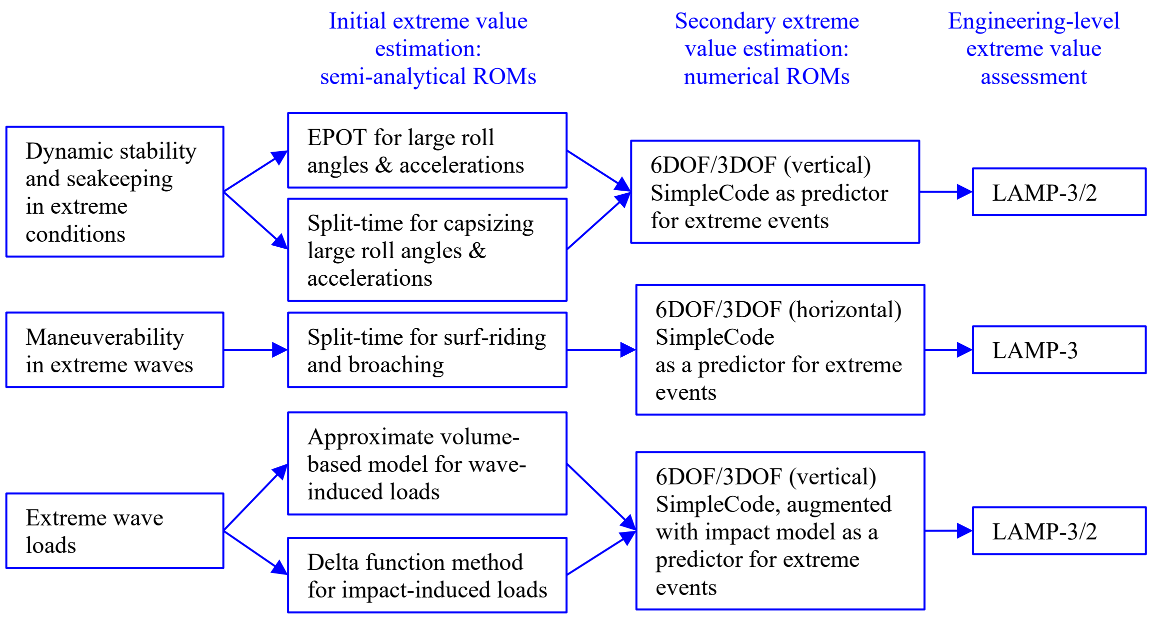

Figure 3: Envisioned Design Application of ROMs for Extreme Events, within a Multi-Fidelity Framework

A simple example of Reed (2021) has shown that these extractions were not necessarily correct – the very largest ROM event did not produced the very largest higher fidelity event. However, it did seem that the high-fidelity model would find an extreme event if it is given a set of conditions where ROM shows its largest responses. In order to reliably use ROMs in such an extreme characterization, and to identify situations in which ROMs cannot be used, it may be necessary to quantify the uncertainty of the ROM for the prediction.

Uncertainty quantification is a part of the extrapolation procedure using the split-time and EPOT methods; see *e.g.* Weems *et al.* (2022) and Campbell *et al.* (2022). The uncertainty addressed by the cited references is of a statistical nature, *i.e.* caused by the finite volume of data used for these estimates

In the example of Reed (2021), high-fidelity and ROM models were run on the same wave records, so differences in the observed outcome should come from differences in assumptions, *i.e.* should be associated with modeling uncertainty.

Uncertainty Quantification with Regression

Regression is presented in Figure 1 as a way to fit the ROMs with high-fidelity data. Regression methods come with uncertainty quantification techniques; see *e.g.* Faraway (2005). Aram and Park (2022) describe the formal application of linear regression and uncertainty quantification to roll decay data. A few key elements of that work are discussed here.

A linear regression equation presents high-fidelity data, referred to as a response vector $\vec{y}$, with the following equation:

$$\vec{y} = \mathbf{X} \cdot \vec{c} + \vec{\varepsilon} \tag{1}$$

Where $\vec{c}$ is a vector of parameters, $\mathbf{X}$ is a matrix of predictors, and $\vec{\varepsilon}$ is a vector of residuals. The approximation with the regression model, which is the ROM, is expressed as:

$$\hat{\vec{y}} = \mathbf{X} \cdot \hat{\vec{c}} \tag{2}$$

where $\hat{\vec{c}}$ is the estimate of $\vec{c}$.

The central assumption of regression modeling is that the difference between the ROM and high fidelity data is caused by random reasons; thus $\vec{\varepsilon}$ is a random vector with zero mean normal distribution. Normality naturally comes from the Central Limit Theorem, as it is assumed that random reasons are many and their contributions are approximately equal.

The vector of parameters $\vec{c}$ is estmated from the condition of the minimum of the sum of the squares of residuals. In the case of linear regression, this leads to an analytical expression:

$$\vec{\varepsilon}\vec{\varepsilon}^T \to \min \Rightarrow \hat{\vec{c}} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\vec{y} \tag{3}$$

where symbol T stands for transposing a matrix. Standard residual error is a measure of the variability of the vector of residuals:

$$\hat{\sigma}^2 = \frac{1}{n-p}\vec{\varepsilon}\vec{\varepsilon}^T \quad (4)$$

where n is the number of dependent variables (number of rows of the matrx **X** as well as the length of the vectors $\vec{y}$, $\hat{\vec{c}}$, and $\vec{\varepsilon}$), and p is the number of predictors (*i.e.* number of columns of the matrix **X**).

The minimum sum of squares calculation is essentially an averaging procedure, so as residuals are normal, the parameters follow a Student's t distribution with n-p degrees of freedom. The boundaries of the confidence interval for the i^{th} parameter are expressed as:

$$\hat{c}_i^{up,low} = \hat{c}_i \pm t_{\alpha/2,n-p}\hat{\sigma}\sqrt{(\mathbf{X}^T\mathbf{X})_{ii}^{-1}} \quad (5)$$

where $t_{\alpha/2,n-p}$ is the quantile to the confidence probability corresponding to α and $\hat{\sigma}^2(\mathbf{X}^T\mathbf{X})_{ii}^{-1}$ is a variance of the estimate of the i^{th} parameter.

If the ROM estimate $\hat{y}_0$ is considered without a residual error, *i.e.:*

$$\hat{y}_0 = \vec{x}_0 \cdot \hat{\vec{c}} \quad (6)$$

where $\vec{x}_0$ is a particular instance of the vector of predictors, its confidence interval can be constructed by treating (6) as a deterministic vector-valued function of random argument $\hat{\vec{c}}$:

$$\hat{y}_0^{up,low} = \hat{y}_0 \pm t_{\alpha/2,n-p}\hat{\sigma}\sqrt{\vec{x}_0^T(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\vec{x}_0} \quad (7)$$

where $\hat{\sigma}^2\vec{x}_0^T(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\vec{x}_0$ is the variance of the estimate (6). The confidence interval (7) describes the uncertainty of the ROM estimate if it is interpreted as a mean estimate; see *e.g.* Faraway (2005).

If a residual error is expected, then the ROM estimate is:

$$\hat{y}_0 = \vec{x}_0 \cdot \hat{\vec{c}} + \varepsilon_0 \quad (8)$$

where ε_0 is an unknown residual error that is assumed to be independent of $\hat{\vec{c}}$. Then the variance of the estimate is $\hat{\sigma}^2\left(1 + \vec{x}_0^T(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\vec{x}_0\right)$ and the confidence interval is expressed as follows:

$$\hat{y}_0^{up,low} = \hat{y}_0 \pm t_{\alpha/2,n-p}\hat{\sigma}\sqrt{1 + \vec{x}_0^T(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\vec{x}_0} \quad (9)$$

Estimate (8) is sometimes referred to as a "future value prediction," and (9) is considered as a prediction interval. Examples of the calculation of the intervals (7) and (9) for the roll decay data, generated with RANS for ONR Topside Series tumblehome configuration, are presented in Aram and Park (2022).

Another example from Aram and Park (2022) is an application of a nonlinear regression, in which a decaying cosine function was fitted to the roll decay data. The nonlinear regression is essentially an optimization problem solved numerically:

$$\hat{\vec{b}} = \mathrm{argmin}(\vec{\varepsilon}\vec{\varepsilon}^T) \quad (10)$$

where $\hat{\vec{b}}$ is a vector parameters of the nonlinear ROM. Nonlinear regression comes with its own uncertainity quantification techniques.

Modeling Uncertainty

The ONR Topsides Series tumblehome configuration considered by Aram and Park (2022) is known for its strong geometric nonlinearity, manifested in the vertical deviation of the backbone curve at small roll angles; see Figure 7 of Aram and Park (2022). The quadratic fit for the logarithmic decrement, which results in a quadratic plus cubic model for damping in the time domain, does not really fit the data. The more flexible decaying cosine curve shows the smallest uncertainty when the large and small roll amplitudes are processed separately. However the "quadratic-plus-cubic" model may be preferable for practical reasons.

In general, this example is meant to illustrate a situation in which a model that should be used in the simulations is not necessarily a "good" model from a data perspective. This type of model can be referred to as a "useful" model.

If we assume there is a way to fit a "good" model and evaluate its uncertainty, then it would be logical to consider the uncertainty of a "good" model as statistical uncertainty caused by random reasons – essentially by the finite volume of data. The difference between the "good" model and the "useful" model may be then associated with modeling uncertainty.

In order to avoid the difficulty of finding a "good" model, a non-parametric regression can be used. In particular, Gaussian Process Regression (GPR) appears to be a good candidate (*e.g.* Rasmussen and Williams, 2006).

The idea of GPR is quite intuitive. The data are assumed to be sampled from a non-stationary stochastic process following Gaussian distribution. The model itself is a mean value function of this process. In order to characterize uncertainty, it is necessary to determine the autocovariance function, after which the Gaussian distribution is completely defined for each x_0.

For the single-value GPR, the mean value function is expressed as:

$$\mu(x_0) = \vec{K}(x_0)(\mathbf{K} + \sigma_n^2 \mathbf{I})^{-1}\vec{y} \qquad (11)$$

where x_0 is a value where the prediction is computed, $\vec{y}$ is a vector of responces consisted from n elements (logarithmic decrement values in the example of Aram and Park, 2022), **I** is an n x n identity matrix, **K** is the n x n covariance matrix, and the vector-valued function $\vec{K}(x_0)$ is defined as:

$$\vec{K}(x_0) = \begin{pmatrix} k(x_0, x_1) \\ \dots \\ k(x_0, x_n) \end{pmatrix} \qquad (12)$$

where $\vec{x} = (x_1, \dots, x_n)^T$ is a vector of predictors (roll amplitudes in the example of Aram and Park, 2022) and $k(x_0, \vec{x})$ is a kernel function defined as:

$$k(x_0, x_i) = \sigma_h^2 \cdot \exp\left(-\frac{(x_0 - x_i)^2}{L}\right) \qquad (13)$$

σ_h and L are hyper parameters that are normally found through an optimization procedure. The covariance matrix **K** is computed with the kernel function as:

$$\mathbf{K} = \begin{pmatrix} k(x_1, x_1) & \cdots & k(x_1, x_n) \\ \vdots & \ddots & \vdots \\ k(x_n, x_1) & \cdots & k(x_n, x_n) \end{pmatrix} \qquad (14)$$

Finally, σ_n is a standard deviation of noise that is found through an optimization procedure along with the hyper parameters.

Figure 4 shows a comparison of GPR vs. linear regression computed for the RANS data from Aram and Park (2022).

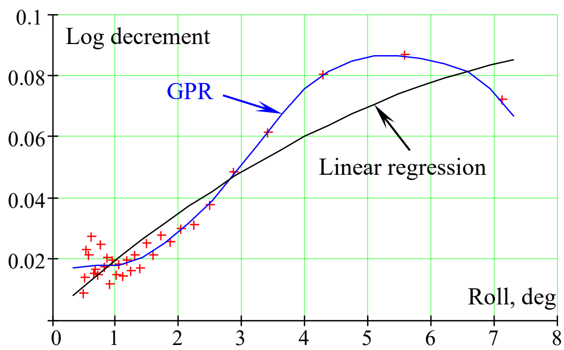

Figure 4: Comparison of GPR and Linear regression for RANS roll decay data, from Aram and Park (2022)

The linear regression was used to fit a quadratic parabola:

$$F(\varphi) = c_0 + c_1\varphi + c_2\varphi^2 \qquad (15)$$

Parameters of linear regression and GPR are shown in Table 1. In this example, the GPR parameters were set manually without applying an optimization procedure.

Table 1: Regression parameters for RANS roll decay data, from Aram and Park (2022)

Parameter	Value	Parameter	Value
c_0	$2.782\cdot10^{-3}$	σ_h	1.0
c_1	0.018	L	1.0
c_2	$-9.214\cdot10^{-4}$	σ_n	0.1

Uncertainty of the GPR is quantified through the generation of instances of a non-stationary Gaussian process with mean value function (11) and the following covariance function:

$$cov(x_0) = k(x_0, \vec{x}) \\ -\vec{K}(x_0)(\mathbf{K} + \sigma_n^2\mathbf{I})^{-1}\left(\vec{K}(x_0)\right)^T \qquad (16)$$

It is not yet clear how exactly to formulate the modeling uncertainty based on the observed difference between the “good” and the “useful” models. It may be necessary to assume that the statistical and modeling uncertainty can be treated as independent random quantities, likely with Gaussian distribution.

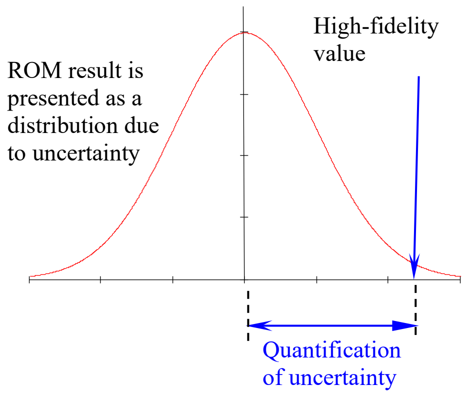

Figure 5: On quantification of ROM uncertainty

The propagation of this uncertainty though a dynamical system is based on consideration of the dynamical system as a deterministic function of random arguments. A likely approach would be to collect the results of a large number of ROM evaluations spanning the uncertainty bands of key parameters and coefficients. The result of this uncertainty propagation could be a distribution of a certain characteristic of interest (say, roll angle value or turning diameter in waves) that can be compared to a high-fidelity result; see Figure 5.

4. SUMMARY AND CONCLUSIONS

This paper focused on Reduced Order Models (ROMs) for ship hydrodynamics and their role in a multi-fidelity modeling framework assessing ship responses including extreme events, though the ideas are intended to be applicable to the broader application of ROMs. There are two objectives of this paper: the first is to review the development of relevant ROMs in an attempt to see if there are some general principles leading to successes. The second objective was to discuss possible uncertainty quantification of ROMs.

The review of the ROMs, while being incomplete, allows the formulation of two general principles that ROM development seems to follow:

- Schematization of hydrostatic and Froude-Krylov forces; schematization of excitation, including parametric excitation.
- Reducing dimensionality of the space of random parameters through regression or /and active sampling.

Uncertainty quantification is an important tool for confident application of ROMs within the multi-fidelity modeling framework. It becomes especially useful when the results from high-fidelity simulations differs from ROMs. Two types of uncertainty were considered:

- Statistical uncertainty caused by random reasons such as finite volume of data;
- Modeling uncertainty caused by necessary simplifications of ROM.

Regression methods come with techniques to quantify uncertainty. However the regression methods are data-driven and assume that the model fits the data. To separate statistical and modeling uncertainty, “good” and “useful” type of models are introduced. The “good” model fits the data well, while the “useful” model is needed for practical reasons. Non-parametric regression such as Gaussian Process Regression (GPR) may be a useful tool for a “good” model. The difference between “good” and “useful” model may be helpful to reveal and quantify modeling uncertainty.

At present, a quantitative evaluation of the accuracy and effectiveness of ROMs is incomplete, though elements are, perhaps, coming into focus. The practical application of ROMs within the multi-fidelity framework is still very much based on engineering judgement. It is hoped that the development of practical approaches to quantify the uncertainty in ROMs will improve the robustness and breadth of their applications in the future.

5. ACKNOWLEDGEMENTS

The work was supported by the US Office of Naval Research (ONR) under the supervision of Dr. Woei-Min Lin. The participation of Prof. Sapsis was facilitated by the NSWCCD Summer Faculty Program, while the participation of Prof. Pipiras was partially facilitated by the NSWCCD Summer Faculty and Sabbatical Programs, both of which were also managed by Dr. Jack Price. Discussions with Dr. Vadim Belenky of David Taylor Model Basin have been very useful.

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