# Numerical Simulation KPI Equation

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**ABSTRACT**

The analysis of different numerical procedures for nonlinear equations describing strong waves evolution is carried out. We have chosen master equation, that is the generalization of Kadomtsev-Petviashvili-I Equation (KPI), that shows major part of the problems in ocean waves evolution and at the same time most difficult from the point of view of numerical algorithm stability. Some indications for choosing of correct numerical procedures are given.

**Keywords:** *Kadomtsev-Petviashvili-I Equation, numerical methods, solution stability*

In the numerical integration of KPI equation instead of the original equation its integral-differential analogue is considered

$$u_t + 0.5(u^2)_x + \beta u_{xxx} = \eta \int_{-\infty}^{x} u_{yy}(x', y, t)dx' + G(x,y) \quad (1)$$

The solution of equation (1) in the half-plane $t \geq 0$ is sought for the initial distribution $u(x,y,0) = q(x,y)$.

Numerical simulation of the equation (1) is carried out using linearized implicit finite-difference scheme, with, in some cases, flux correction technique (FCT).

Solution of the equation (1) is performed using the approximation for the central-difference operators. The order of approximation of a difference scheme in the calculation is of the order of $O(\Delta t, \Delta x^2, \Delta y^2)$. The resulting system of difference equations is reduced to the form:

$$a_j \Delta u_{j-2,k}^{n+1} + b_j \Delta u_{j-1,k}^{n+1} + c_j \Delta u_{jk}^{n+1} + d_j \Delta u_{j+1,k}^{n+1} + e_j \Delta u_{j+2,k}^{n+1} = f_{jk}^n \quad (2)$$

with $\Delta u_{jk}^{n+1} = u_{jk}^{n+1} - u_{jk}^n$.

The system (2) is solved by the five-point sweep (Thomas algorithm).

At the boundaries of the computational domain $[x_1,\ x_M] \times [y_1,\ y_L]$ set of difference boundary conditions is imposed. Traditionally the so-called "flow conditions" are used: $u_x = u_{xx} = 0$ along the boundary lines $x_1$ and $x_M$, and $u_y = 0$ along the lines $y_1$ и $y_L$.

As the initial distributions three surfaces were selected:

1. The parallelepiped.
2. Gaussian distribution.
3. The ellipsoid of rotation.

In our case, we want to investigate the influence of the shape of the initial distribution on the further evolution of the perturbation. To unify the choice of distribution parameters, we fix the volume and variety of shapes and parameters for ellipses that fit into the bottom of the box.

Compare the numerical calculation results with the known analytical solution of the KPI equation.

We apply the finite-difference scheme (2) for the equation, similar to (1):

$$[u_t + 3(u^2)_x + u_{xxx}]_x = 3u_{yy} \quad (3)$$

For the equation (3) there exist lump type soliton solution, i.e. in the form:

$$u(x,y,t) = 4\frac{-(x-3\mu^2 t)^2 + \mu^2 y^2 + 1/\mu^2}{[(x-3\mu^2 t)^2 + \mu^2 y^2 + 1/\mu^2]^2} \quad (4)$$

On fig. 1 we compare the exact solution with the numerical solution for a single point in time when $y = 0$.

One can see the results difference is within the tolerance accepted for purely implicit difference scheme.

In KdVB equation is calculated using a difference scheme, which includes a flux correction procedure. It is interesting to examine the possibility of the use of this approach in our case.

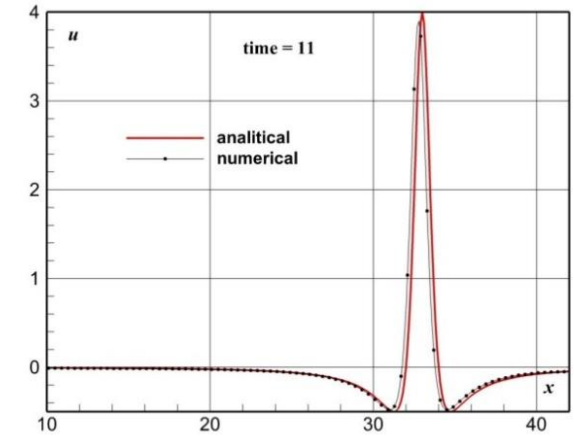


**Figure 1: Comparison of exact solution (4) with numerical one for $t = 11$. Mesh being $700 \times 500$, $\Delta x = \Delta y = 0.1$, $\Delta t = 2 \cdot 10^{-5}$, $y = 0$.**

Finally, after the analysis carried out after numerical experiments, it was decided not to use, in general, anti-aliasing algorithm. The resulting numerical dispersion ripples did not significantly affect the nature of the perturbations and, most importantly, do not underestimate the amplitude and velocity of the soliton.

Let us consider the dependence of the results of the calculation on the initial distribution. To do this, some of the values of geometrical parameters are necessary to be fixed. The volume of initial perturbation is the same for all figures: $V = 120$.

Calculations were carried out without smoothing procedure up to the time $t = 8$; the number of nodes is $800 \times 700$; the time step $\Delta t = 5 \cdot 10^{-5}$; mesh steps $\Delta x = \Delta y = 0.1$.

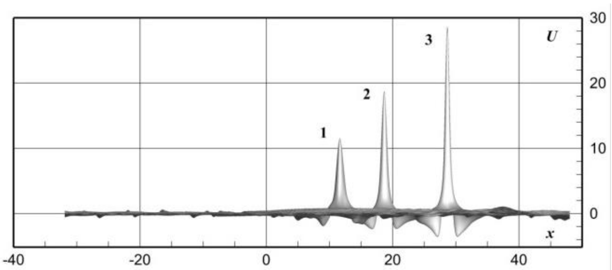


**Figure 2: The formation of solitons with different initial distributions for $t = 8$, $V = 120$. Mesh $800 \times 700$, $\Delta x = \Delta y = 0.1$, $\Delta t = 5 \cdot 10^{-5}$, $y = 0$.**

As it clearly seen from fig. 2 the largest soliton is formed from the original form of the ellipsoid of revolution.

Consider the initial distribution of Gaussian type, with different volumes. All calculations were performed without anti-aliasing. With the help of numerical simulation we find the situation in which after relatively small increase in volume, compared with the previous value, sharply increases the amplitude of the resulting soliton. The process is similar to the pressure jump (fig. 3).

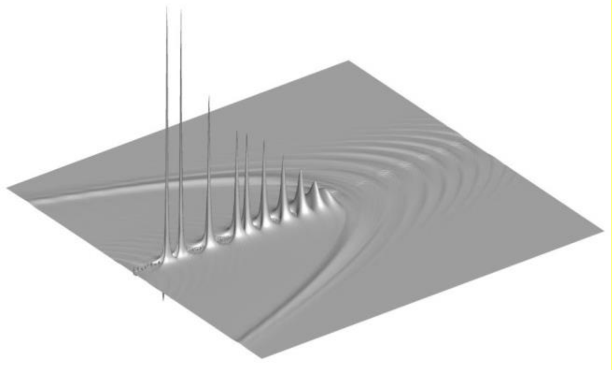

**Figure 3: 3D demonstration of an abrupt increase in the soliton amplitude for the initial conditions of the Gaussian form at $t = 7$. Mesh $500 \times 500$, $\Delta t = 10^{-4}$, $\Delta x = \Delta y = 0.2$.**

Some problems may appear when the source in rhs is switched on. We have selected a source in the form of an ellipsoid of revolution, as in the case 3 of the initial distribution. Calculations of the equation (1) with a source, a natural analogue of the impact on the water surface, provide numerous options of possible situations with formation of large-amplitude solitons. The source itself generates solitons. Source intensity varies in a wide range. Field exposure source is limited by the natural conditions, but eventually forms a cluster of perturbations, out of which solitons of different amplitudes are formed. For example, we present the evolution of the perturbation without taking into account the initial distribution of any type (see Fig.4).

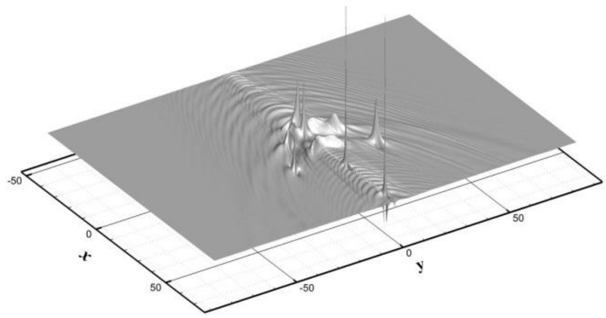


**Figure 4: 3D perturbation generated by a source at $t = 15.5$. Mesh $600 \times 850$, $\Delta t = 10^{-4}$, $\Delta x = \Delta y = 0.2$.**

## CONCLUSIONS

Some indications for choosing of correct numerical procedures from our study can be formulated as follows

1. The proposed scheme has a sufficient resolution for areas with large gradients.

2. Our approach effectively describes the process of soliton formation and propagation with their characteristics preservation.

3. That scheme satisfactorily calculates cases with initial distributions that are not completely integrable.

4. The time step strongly depends on the initial distribution, since the evolution of the perturbation leads to a velocity in the order of magnitude greater than is seen with a linear analog of KPI equation

5. Using of the smoothing procedure leads eventually to an underestimation of the amplitudes of the solitons. The need for a FCT procedure is not obvious.

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