On the Tail of Nonlinear Roll Motions

Vadim Belenky, *Naval Surface Warfare Center Carderock Division*
Dylan Glozter, Vladas Pipiras, *University of North Carolina*
Themistolkis P. Sapsis, *Massachusetts Institute of Technology*

ABSTRACT

The paper describes the qualitative study of the tails of the distribution of large-amplitude roll motions. The nonlinearity of a dynamical system is modeled with piecewise linear stiffness with stable and unstable equilibria. Closed-form formulae were derived for the peak value and its distribution. The tail of the distribution is heavy until in close proximity to the unstable equilibrium and then becomes light with the right bound at the unstable equilibrium. It is shown that the tail structure is related to the shape of the stiffness curve. Physical reasoning for such tail structure is based on the phase plane topology. The tail first becomes heavy due to stretching of the phase plane, which is a result of nonlinearity. The inflection point in the tail (when it becomes light) is related to increased capsizing probability in the vicinity of unstable equilibrium; the position of the inflection point can be evaluated, defining domain of heavy tail applicability.

Keywords: *Nonlinear Roll Motion, Distribution, Extremes*

1. INTRODUCTION

Probabilistic assessment of partial dynamic stability failure is essentially an extreme value problem for nonlinear roll motions. Some progress has been recently reported by Campbell, *et al* (2016) on applying Generalized Pareto distribution (GPD) to model the extreme values of roll peaks, above appropriate threshold (Coles, 2001). Mathematical aspects of the problem are treated in (Glotzer et al 2016). Statistical validation of this method was described by Smith and Zuzick (2015). While, in general, the method has shown satisfactory performance, its accuracy may be improved by applying one-parameter GPD instead of two-parameter GPD. It requires introducing a relation between the GPD parameters based on physical properties of the dynamical system. This relation is the main objective of this paper.

Normally, GPD has two parameters: shape and scale. If the shape parameter equals zero, GPD turns into the exponential distribution. This is the case of a normally distributed quantity; distribution of its extreme values can be approximated by the exponential distribution. If the shape parameter is positive, the tail is usually referred to as "heavy," as its probability of extreme value is higher compared to normal/exponential case. If the shape parameter is negative, the probability of extreme value is lower compared to exponential and the tail is referred as "light." One of the specific features of a light tail is a right bound, the upper limit of the distribution; all values exceeding the right bound have zero-probability.

The appearance of right bound in a distribution of roll peaks has a clear physical reason. A peak implies a return after reaching a local maximum. As a ship may capsize, there is a limit for the roll peak, which should be reflected as a right bound by statistics. However, GPD fitting, reported in Campbell, *et al* (2016) resulted in positive shape parameter and no right bound.

The question this paper tries to answer formulates as follows: if a ship can capsize, the tail of roll peak distribution should be light, so why is a heavy tail observed in numerical simulations?

2. PIECEWISE LINEAR SYSTEM

A dynamical system with piecewise linear stiffness is probably the simplest model of capsizing, as it allows recreation of correct phase plane topology, see Figure 1. It also allows a closed form solution for probability of capsizing under some assumptions; see review in (Belenky, *et al* 2016). So consider a dynamical system:

$$\ddot{\phi} + 2\delta\dot{\phi} + \omega_0^2 f_L(\phi) = f_{E\phi}(t) \qquad (1)$$

where δ is a linear damping coefficient and $f_{E\phi}$ is a stochastic process of roll excitation, while the roll stiffness f_L is shown in Figure 1. It is assumed that the excitation is "switched-off" once the roll angle exceeds ϕ_{m0}, reflecting absence of resonance for negative *GM* and limited ability to react on waves.

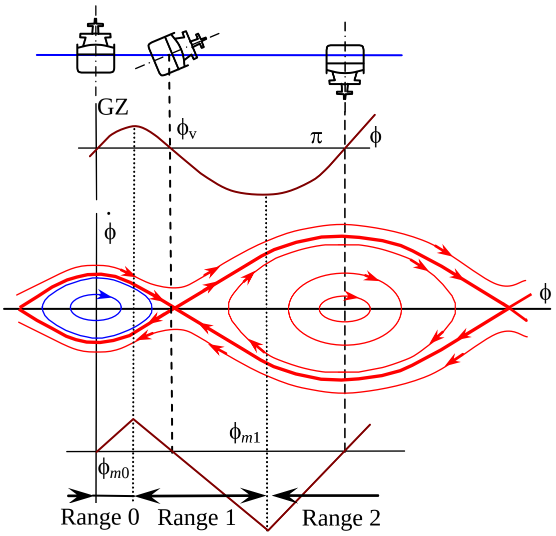

Figure 1 Phase plane topology of capsize and piecewise linear stiffness (Belenky, et al 2016).

Here absence of damping for $\phi>\phi_{m0}$ is also assumed. While in reality roll damping is increased for large roll angles, it is not expected to cause qualitative change. In the absence of capsizing, the solution for the range 1 is expressed as:

$$\phi = H\cosh(\omega_1 t+\varepsilon)+\phi_v \qquad \phi_{m0}<\phi\le\phi_{m1} \tag{2}$$

where: $\omega_1=\omega_0\sqrt{k_1}$, arbitrary constants are

$$H=-\frac{1}{\omega_1}\sqrt{\omega_1^2(\phi_{m0}-\phi_v)^2-\dot{\phi}_1^2} \tag{3}$$

$$\varepsilon=\tanh^{-1}\left(\frac{\dot{\phi}_1}{\omega_1(\phi_{m0}-\phi_v)}\right) \tag{4}$$

$\dot{\phi}_1$ is a roll rate at upcrossing.The value of peak is expressed as:

$$\phi_{\max}(\dot{\phi}_1)=H+\phi_v \qquad 0<\dot{\phi}_1\le\dot{\phi}_{cr} \tag{5}$$

$\dot{\phi}_{cr}$ is critical roll rate corresponding to capsizing conditions:

$$\dot{\phi}_{cr}=\omega_1(\phi_v-\phi_{m1}) \tag{6}$$

3. DISTRIBUTION OF PEAKS

Formula for the peak (3) is a deterministic function of a single random variable. This random variable is the roll rate at the instant of upcrossing. Assuming that the upcrossings of the level ϕ_{m0} are so rare, that is the upcrossing events can be assumed indendent, then the distribution of the roll rate at upcrossing follows Rayleigh (see Leadbetter, *et al.* 1983, Lindgren, 2013, also in Belenky, *et al.* 2016):

$$f_d(\dot{\phi}_1)=\frac{\dot{\phi}_1}{\sigma_d^2}\exp\left(-\frac{\dot{\phi}_1}{2\sigma_d^2}\right) \qquad \dot{\phi}_1>0 \tag{7}$$

Where σ_d is a standard deviation of roll rates for range 0, *i.e.* without influence of crossings. To ensure that only roll peaks are considered, there is no capsizing and a normalizing constant is needed. It is equal to probability of capsizing:

$$f_d(\dot{\phi}_1)=\left(1-\exp\left(-\frac{\dot{\phi}_{cr}^2}{2\sigma_d^2}\right)\right)^{-1}\frac{\dot{\phi}_1}{\sigma_d^2}\exp\left(-\frac{\dot{\phi}_1^2}{2\sigma_d^2}\right) \qquad 0<\dot{\phi}_1<\dot{\phi}_{cr} \tag{8}$$

The function (5) is monotonic; the distribution of this function is:

$$f_{\max}(\phi_{\max})=f_d\left(G^{-1}(\phi_{\max})\right)\cdot\left|\frac{d}{d\phi_{\max}}G^{-1}(\phi_{\max})\right| \qquad \phi_{m0}<\phi_{\max}<\phi_v \tag{9}$$

where $G^{-1}$is an inverse of the function (5)

$$G^{-1}(\phi_{\max})=\omega_1\sqrt{(\phi_v-\phi_{m0})^2-(\phi_v-\phi_{\max})^2} \qquad \phi_{m0}<\phi_{\max}<\phi_v \tag{10}$$

Substitution of (10) and (8) into (9) yields the following distribution density.

$$f_{\max}(\phi_{\max})=\frac{\phi_v-\phi_{\max}}{C}\exp\left(-\frac{\omega_1^2(\phi_v-\phi_{\max})^2}{2\sigma_d^2}\right) \qquad \phi_{m0}<\phi_{\max}<\phi_v \tag{11}$$

where

$$C=\frac{\sigma_d^2}{\omega_1^2}\left(\exp\left(-\frac{\omega_1^2(\phi_v-\phi_{m0})^2}{2\sigma_d^2}\right)-1\right) \tag{12}$$

Distribution (11) is plotted in Figure 2.

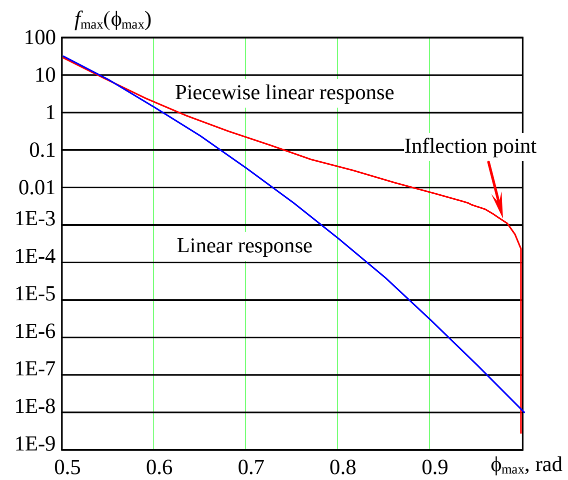

Figure 2 Distribution of peaks of piecewise linear and linear response

To see if the distribution (11) has a heavy tail, compare it to the distribution of peak of a linear system, corresponding to the range 0. Distribution of large peaks of a linear system can be approximated with truncated Rayleigh distribution (Belenky and Campbell 2012):

$$f_{L\max}(\phi_{\max}) = C_L \frac{\phi_{\max}}{\sigma_x^2} \exp\left(-\frac{\phi_{\max}^2}{2\sigma_x^2}\right)$$
$$\phi_{\max} > \phi_{m0} \qquad (13)$$
$$C_L = \exp\left(\frac{\phi_{m0}^2}{2\sigma_x^2}\right)$$

The tail of Rayleigh distribution can be approximated with exponential distribution; thus equation (13) may serve as benchmark; a larger probability than (13) means heavy tail. Figure 2 shows that the piecewise linear system produces this heavy tail through practically the entire range 1. Then, it reaches an inflection point and quickly tends to zero.

Figure 2 answers the question, posed at the end of section 1. The tail actually is heavy for most of the interval and then turns light in the vicinity of unstable equilibrium.

Peaks of the response of the piecewise linear system with unstable equilibrium shows an interesting behavior. The "true" limiting distribution has a light tail, but a heavy tail can be used for approximation at least until the "inflection point." One may say that the piecewise linear system has two tails. What are the conditions for having two tails?

4. DEPENDENCE ON THE SECOND SLOPE

Consider behavior of the distribution (11) when the slope coefficient k_1 tends to zero. Using the relation between k_1 and the position of unstable equilibrium

$$\phi_v = \phi_{m0} \frac{1+k_1}{k_1} \qquad (14)$$

When k_1 reaches zero, the unstable equilibrium ceases to exist.

$$\lim_{k_1 \to 0} \phi_v = \lim_{k_1 \to 0} \left(\phi_{m0} \frac{1+k_1}{k_1}\right) = \infty \qquad (15)$$

The limit transition converts equation (11) into the exponential distribution:

$$\lim_{k_1 \to 0} f_{\max}(\phi_{\max}) = \frac{\omega_1^2 \phi_{m0}}{\sigma_d^2} \times \exp\left(-\frac{\omega_1^2 \phi_{m0}}{\sigma_d^2}(\phi_{\max} - \phi_{m0})\right) \qquad (16)$$

The process of this limit transition is illustrated below. The slope of the range 1 is changed systematically from -1 to 0, as plotted in Figure 3. Figure 4 shows corresponding changes in the distribution of the peaks. The heavy part of the tail becomes lighter, until it reaches the exponential distribution (16) for k_1=0. The "inflection point" moves to the right, until it eventually disappears when the position of unstable equilibrium goes to infinity.

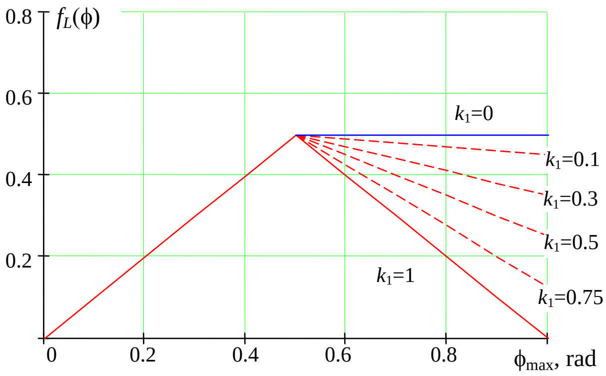

Figure 3 Variation of piecewise linear stiffness

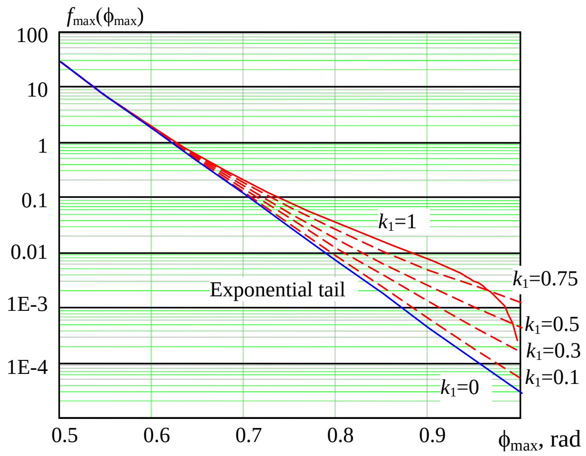

Figure 4 Distribution of peaks of pricewise linear response for different slopes of the second range

The changes in the slope coefficient for the range 1 mean changes in the shape of stiffness. Thus, shape of the stiffness after the maximum defines the shape of the tail, while the position of the unstable equilibrium defines the position of the "inflection point." The softening nonlinearity (k_1>0) seems to be responsible for the "two-tails" structure. It disappears when k_1 becomes zero.

5. WHITE NOISE EXCITATION

The relation between “two-tails” structure, shape of stiffness and presence/absence of the unstable equilibrium points to a possible fundamental relation between the distribution and topology of the phase plane. This link may be revealed if one gets a closed-form expression for joint distribution of motions and velocities. It can be done using the Kolmogorov-Fokker-Plank equation, if white noise excitation is assumed. Indeed, it is far from reality, but the system (1) under white noise excitation may have similar relation between the distribution and phase plane.

Assuming

$$f_{E\phi}(t) = s\dot{W}(t) \tag{17}$$

where W(t) is Wiener process and s its scaling factor or intencity. The the steady-state joint distribution of the motions and velocities is expressed as (see *e.g.* Sobczyk, 1991):

$$f_{\infty}(\phi,\dot{\phi}) = C_W \exp\left(-\frac{4\delta}{s^2} H(\phi,\dot{\phi})\right) \tag{18}$$

where C_W is a normalizing constant and *H(..)* is the Hamiltonian (total energy without disspation) of the dynamical system (1)

$$H(\phi,\dot{\phi}) = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{19}$$

Potential $V(\phi)$ is symmetrical relative to the origin and for $\phi < \phi_v$ is expressed as:

$$V(\phi) = \frac{\omega_0^2}{2}\begin{cases}\phi^2 & 0 \le \phi < \phi_{m0} \\ \phi_{m0}^2 - k_1(\phi - \phi_{m0})^2 & \phi_{m0} \le \phi\end{cases} \tag{20}$$

The current study is focused on the properties of the tail of large-amplitude response, so the distribution (18) needs to be limited to non-capsizing case. In terms of phase plane, it corresponds only to the area within the separatrix, see Figure 5. The Hamiltonian implicitly contains definition of the separatrix, as it is the only line going through the unstable equilibria:

$$\dot{\phi}_s(\phi) = \pm\sqrt{H(\pm\phi_v, 0) - V(\phi)} \tag{21}$$

The distribution of piecewise linear response, not leading to capsizing is expressed as:

$$f_s(\phi) = C_S \int_{-\dot{\phi}_s(\phi)}^{\dot{\phi}_s(\phi)} f_{\infty}(\phi,\dot{\phi})\,d\dot{\phi} \tag{22}$$

where C_S is another normalizing constant.

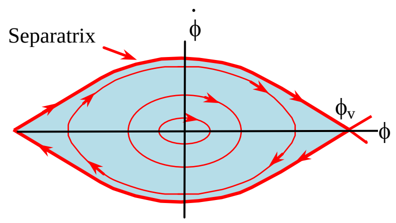

Figure 5 Separatrix and non-capsizing area

Figure 6 shows distribution of piecewise linear response under the “no-capsize” condition computed with formula (22). This distribution has three distinct regions: Gaussian core (i), heavy tail (ii) and light tail (iii). The structure of the tail is exactly the same as in the previous case in Figure 2, where excitation was correlated but switched off above the knuckle point (where damping was absent, too).

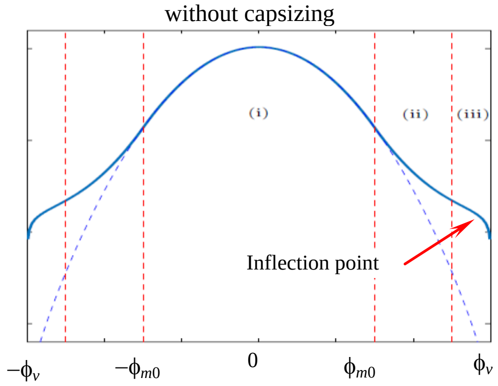

Figure 6 Distribution of piecewise linear response under no-capsize condition

The result in Figure 6, is one more argument that the structure of the tail is defined by stiffness shape. Thus, the correlation of the excitation can be neglected for this type of qualitative study. This provides a number of research tools that can only be applied for white noise excitation.

6. HEAVY TAIL STRUCTURE

Two previous sections presented some arguments that the observed tail structure is a result of the stiffness shape, and presence of the unstable equilibrium, in particular. Presence of the unstable equilibrium makes nonlinearity soft. The piecewise linear system does not differ much in that sense from a nonlinear system with smooth stiffness, as most known qualitative properties are present (Belenky, 2000).

Figure 7 compares linear and piecewise / nonlinear systems both in terms of potential and phase plane. As the linear system contains more potential energy, the potential function of the piecewise system is always below the linear one. The phase trajectories are, in fact, the level lines of the potential function. As a result the phase plane of the piecewise linear system (1) or a system with softening nonlinear stiffness is stretched compared to a linear system.

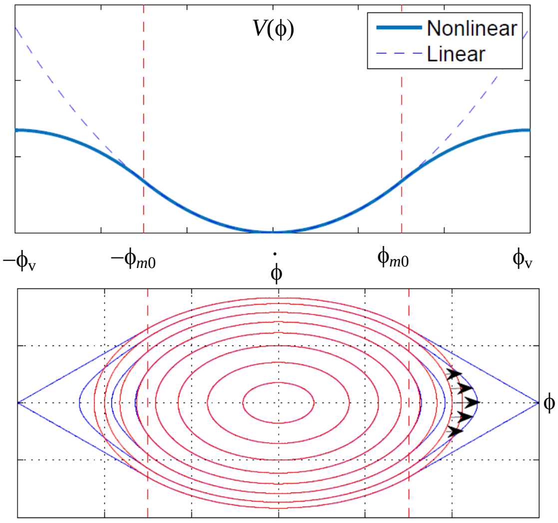

Figure 7 Stretching of the phase plane caused by soft nonlinearity of stiffness

Another way to illustrate this stretching is to compare short portions of time history of the piecewise linear system. Both responses start at the same time instant at the "knuckle" point with the same initial velocity. As it can be seen from Figure 8, the response of the piecewise linear system (2) is always above the similar linear response.

This also means that the piecewise linear system spends more time above the knuckle point that the linear system under the same initial conditions. As a result, probability of finding the piecewise linear system above the knuckle point is higher and the tail of the response is heavier than the linear one.

Also, one can see at Figure 8 that the local maximum of the piecewise linear response is larger than the linear one. Thus the tail of peaks of the nonlinear response is heavier than the linear one, as it can be seen in Figure 2.

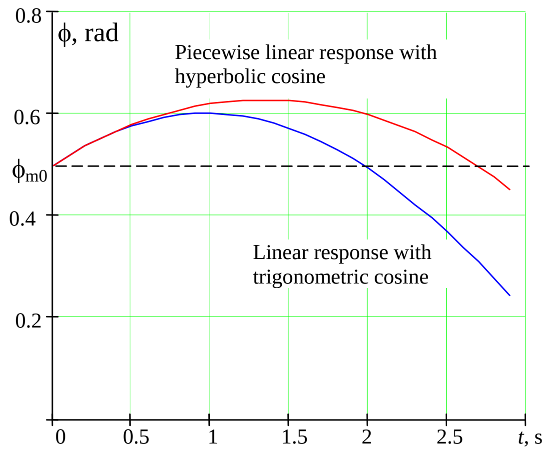

Figure 8 Piecewise linear response above the knuckle point vs. linear response

7. LIGHT TAIL STRUCTURE

Obviously, the tail of both response and its peaks becomes light because of the presence of unstable equilibrium. Consider how it is reflected in the distribution (22), by substitution formula (18) and (19):

$$f_s(\phi) = C_S C_W \exp\left(-\frac{4\delta}{s^2} V(\phi)\right) \times \int_{-\dot{\phi}_s(\phi)}^{\dot{\phi}_s(\phi)} \exp\left(-\frac{4\delta\dot{\phi}^2}{2s^2}\right) d\dot{\phi} \qquad (23)$$

The integrand is in fact the normal distribution as:

$$\sigma_d^2 = \frac{s^2}{4\delta} \qquad (24)$$

The integral term does not play much of a role when the motion displacement is far from the unstable equilibrium. The separatrix goes through very large velocities for most of range 2 and the integral in (23) is close to one (after being multiplied by its normalizing constant). Once the motion approaches the unstable equilibrium, the limits of integration do get close to each other. As a result the integral term in (23) decreases and forces the entire distribution down, until it reaches zero at the point of unstable equilibrium.

Understanding of this mechanism allows estimation of the position of the inflection point. It can be checked that the logarithm of the distribution (11) has the inflection point at:

$$\phi_{\inf} = \phi_v - \frac{\sigma_d}{\omega_1} \qquad (25)$$

Position of the inflection point defines a boundary of the heavy tail range and it should be possible to find it for a general nonlinear system.

8. CONCULSIONS

The original motivation for this study was to answer a simple question, why the GPD fit shows a heavy tail for peaks of roll motions, when it is expected to be light because a ship can capsize and the peaks cannot exceed a certain limit. The answer was found by analyzing a dynamical system with an unstable equilibrium and piecewise linear stiffness. The distribution has “two-tails” structure: it is heavy at first, but becomes light in close vicinity of the unstable equilibrium.

This “two-tails” structure is a result of the presence of an unstable equilibrium and related softening nonlinear stiffness. The heavy tail is a result of stretching of the phase plane. The light tail appears in close vicinity to the unstable equilibrium, where most trajectories lead to capsizing so the probability of non-capsizing is very small.

The “inflection point” of the tails is the boundary between heavy and light tail. Its position can be found and used as a limit of applicability of the heavy tail assumption.

The shape of stiffness and related topology of the phase plane is the main factor defining the tail structure of the response of dynamical system. Qualitative tail structure seems to be the same for the dynamical system with correlated or white noise excitation.

Further research includes a wider variety of nonlinear dynamical systems, as well as metrics of likelihood of capsizing and broaching-to. A technique for estimation of the position of the “inflection point” should be developed for generic nonlinear systems and eventually use this information to reduce uncertainty of GPD fit.

9. ACKNOWLEGEMENT

The work described in this paper has been funded by the Office of Naval Research under Dr. Thomas Fu and by NSWCCD Independent Applied Research (IAR) program. Particiaption of Prof. Pipiras and Prof. Sapsis was was facilitated by the Summer Faculty Program supported by ONR and managed by NSWCCD under Dr. Jack Price, who also managed IAR program. Particpation of Mr. Glozter was facilated by NWCDD NREIP program managed by Ms. Rachel Luu.

10. REFERENCES

Belenky, V. L., (2000) “Piecewise linear approach to nonlinear ship dynamics”, in Contemporary Ideas on Ship Stability Vassalos, D., Hamamoto, M., Papanikolaou, A. and D. Molyneux, eds., Elsevier, pp.149-160.

Belenky, V. and Campbell, B. (2012) “Statistical Extrapolation for Direct Stability Assessment”, *Proc. 11th Intl. Conf. on Stability of Ships and Ocean Vehicles STAB 2012*, Athens, Greece, pp. 243-256

Belenky, V., Weems, K. and K. Spyrou (2016) “Towards a Split-Time Method for Estimation of Probability of Surf-Riding in Irregular Seas”, *in Ocean Engineering* (in Press DOI: 10.1016/j.oceaneng.2016.04.003).

Campbell, B., Belenky, V. and V. Pipiras (2016) “Application of the Envelope Peaks over Threshold (EPOT) Method for Probabilistic Assessment of Dynamic Stability, in Ocean Eng. (DOI: 10.1016/j.oceaneng.2016.03.006).

Coles, S., 2001 An Introduction to Statistical Modeling of Extreme Values. Springer, London.

Glotzer, D., Pipiras, V., Belenky, V., Campbell, B., T. Smith, (2016) "Confidence Interval for Exceedance Probabilities with Application to Extreme Ship Motions", REVSTAT Statistical J. (Accepted).

Leadbetter, M. R., Lindgren, G. & Rootzen, H. (1983), Extremes and Related Properties of Random Sequences and Processes, Springer Series in Statistics, Springer-Verlag, New York-Berlin

Lindgren, G. (2013), Stationary Stochastic Processes, Chapman & Hall/CRC Texts in Statistical Science Series, CRC Press, Boca Raton, FL. Theory and applications

Smith, T. C., and Zuzick, A., (2015) “Validation of Statistical Extrapolation Methods for Large Motion Prediction” in *Proc.* 12th Intl. Conf. on Stability of Ships and Ocean Vehicles (STAB 2015*)*, Glasgow, UK.

Sobczyk, K. (1991), Stochastic Differential Equations, Kluwer Academic Publishers, Dordrecht, The Netherlands.