New models of irregular waves—way forward

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ABSTRACT

In a series of papers, Degtyarev and Reed have presented the theory and provided the results from an autoregressive model for representing a seaway—at a point in space, over a line and over a plane, all as a function of time (1-D, 2-D & 3-D, respectively). In several other papers, Degtyarev and Gankevich have provided the theory for a technique for efficiently computing the velocity potential beneath a prescribed 1-D or 2-D surface, varying with time. Together this series of papers provides the information needed to compute the fully nonlinear hydrostatic and Froude-Krylov pressures under a seaway in an efficient manner without having to be concerned with the computing-time constraints imposed by the use of a Fourier series representation of a seaway imposed by the use of a Longuet-Higgins model. The next step is to apply these models in a seakeeping code so that the practical aspects of using these appealing theoretical approaches can be assessed. This paper provides a very brief description of the methods, and outlines some of the issues that must be dealt with in interpreting them.

KEYWORDS

Autoregressive modelling; Wave modelling; Sea state modelling

1 INTRODUCTION

The, Longuet-Higgins' Fourier series based model of a seaway (Longuet-Higgins, 1962) is distinguished by its clarity and the simplicity of the computational algorithm. However, it is not without some serious shortcomings inherent in models of this class:

- The Longuet-Higgins' model is only designed to represent a stationary Gaussian field. Normal distribution of the simulated process is a consequence of the central limit theorem. Its application to the analysis of more general problems such as the evolution of ocean waves in a storm, or the study of ocean waves distorted in shallow water represents a significant challenge.
- Models of this class are periodic and need a very large number of frequencies in order to generate statistically independent non-repeating waves for long simulations (Belenky, 2005) and the computation time increase linearly with the number of frequencies.
- In the numerical implementation of the Longuet-Higgins' model, it appears that

the rate of statistical convergence is very slow. This is seen as a distortion of the energy spectrum of the simulated process.

- The Longuet-Higgins model is not obviously appropriate when simulating complex waves that have a broad spectrum with many peaks, and in describing extreme events.

These latter three points become particularly critical in numerical simulation. In a time domain computation of the responses of a vessel in a random seaway, the repeated evaluation of the velocity at hundreds or thousands of points on the hull for thousands or tens of thousands of time steps can become a major factor determining the execution speed of the code (Beck & Reed, 2001). This becomes an even more significant issue in a nonlinear computation where the wave model is even more complex. Developing a less time intensive method for modeling the ambient ocean-wave environment has the potential for significantly speeding up the total simulation process.

2 AN AUTOREGRESSIVE MODEL OF OCEAN WAVES

The autoregressive model (ARM) of ocean waves is an alternative to the Longuet-Higgins' approach that models a stochastic moving surface as a linear transformation of white noise with memory. ARMs are commonly used in other areas of probabilistic mechanics and dynamics to model stationary ergodic Gaussian random processes with given correlation characteristics (Box, *et al.*, 2008), but they have not been extensively applied to wind waves.

2.1 One dimensional Wind-Wave Model

The formal mathematical framework of regressive wave models was developed by Spanos (1983), Gurgenidze & Trapeznikov (1988) and Rozhkov & Trapeznikov (1990). The latter built a one-dimensional model of ocean waves $\zeta(t)$, on the basis of an autoregressive-moving average (ARMA) model

In practice, it has been more common to use an autoregressive model:

$$\zeta_t = \sum_{i=1}^{N} \Phi_t \zeta_{t-i} + \epsilon_t, \tag{1}$$

where ζ_t is the wave elevation at time t, N is the order of the model, Φ_i are the regression coefficients, ζ_{t-i} are the N last realizations of $\zeta_t, [i = 1, \ldots, N]$, ϵ_t is Gaussian white noise with variance σ_ϵ^2. The equation for ζ_t can be directly related to the power spectrum of the seaway by:

$$S_\zeta(\omega) = \frac{\sigma_\epsilon^2}{2\pi} \frac{\Delta}{\left|1 + \sum_{j=1}^{N} \Phi_j \exp[-ij\Delta\omega]\right|^2}, \tag{2}$$

where Δ is the sampling interval of the series.

The autoregressive coefficients of (1) can be estimated from the autocovariance function (K_ζ) by solving the Yule-Walker equations:

$$K_\zeta(i) = \sum_{k=0}^{N} \Phi_k \, K_\zeta(k - i), \tag{3}$$

and the variance of the white noise σ_ϵ^2 can be calculated as:

$$\sigma_\epsilon^2 = V_\zeta - \sum_{j=0}^{N} \Phi_j K_\zeta(j). \tag{4}$$

where V_ζ is the variance of the waves being simulated. The derivation of these formulae can be found in Degtyarev & Reed (2011).

In theory, the number of autoregressive coefficients N tends to infinity. In practice, it has been found that remarkably few coefficients are required to recreate the wave surface and to recover the stochastic properties of the wave. As the periodicity of the wave evaluation is dependent only on the random number generator, very long wave records can be modeled without self-repeat and at very small cost.

2.2 3-D Wave Model

For application to numerical simulation in three dimensions (2-D space + 1-D temporal) having components (x, y, t), the expression for the wave elevation is:

$$\begin{aligned}\zeta(x,y,t) &= \sum_{ix=0}^{Nx}\sum_{iy=0}^{Ny}\sum_{it=0}^{Nt}\Phi_{(ix,iy,it)} \\ &\times \zeta(x-ix\cdot\Delta x, y-iy\cdot\Delta y, t-it\cdot\Delta t) \\ &+ \sigma_\epsilon^2\epsilon_{(ix,iy,it)}\end{aligned} \qquad (5)$$

Degtyarev & Boukhanovsky (2000) present numerical procedures for estimating the parameters of the 3-D ARM for waves and the dispersion of the corresponding field of white noise, as well as the transition to a wave field with an arbitrary distribution. The procedures generally follow the one-dimensional implementation and are based on the solution of the generalized Yule-Walker equations (*cf.*, Degtyarev & Reed, 2011), though with additional computational features.

3 IMPLEMENTATION OF THE AUTOREGRESSIVE WAVE MODEL IN A SIMULATION CODE

A principal objective of the current effort is to apply the autoregressive incident wave model to time domain ship motion simulations. The issues and procedures are relevant to any hydrodynamic code; and, to a large degree, the use of autoregressive wave models in general.

In the seakeeping calculations, the following incident wave quantities must be computed:

- Incident elevation at points on the hull surface in order to determine the incident wave waterline and create a panel model of the wetted hull surface
- Incident wave pressure ($\rho\partial\Phi_0/\partial t$) on each wetted hull panel to calculate Froude-Krylov forces
- Incident wave velocity ($\nabla\Phi_0$) at the control point of each body panel for potential flow body boundary condition
- Incident wave velocity ($\nabla\Phi_0$) for the inflow to external forces models such as appendage lift and drag.

In calculations using the standard Longuet-Higgins' model, the incident wave is defined by a discrete set of component waves, each with a specified frequency, amplitude, heading, and phase; and these incident wave quantities are generally computed directly using Fourier series expressions.

With the autoregressive wave model, the incident wave is defined by a regression order (N_x, N_y, N_z) and increment $(\Delta x, \Delta y, \Delta z)$, a set of regression coefficients ($\Phi_{(ix,iy,it)}$), corresponding variance of white noise (σ_ϵ^2) and a set of seeds for the pseudo-random number generator. At each time step of the simulation, the incident wave model is set up by the following steps:

1. Compute the elevation field on a grid of points around the ship
2. Estimate derivatives of the elevation in time and space
3. Solve for the velocity potential field beneath this elevation grid
4. Estimate derivatives of the velocity potential in time (Froude-Krylov pressure) and space (incident wave velocity)
5. Set up interpolation functions for the elevation and potential derivatives on the local grids.

The required evaluations of the incident wave elevation, pressure, and velocity are then handled by the interpolation functions. These steps are described in more detail below.

4 INCIDENT WAVE ELEVATION FIELD

The form of the expression for the autoregression wave elevation (5) naturally leads to the evaluation of the local wave elevation field on a grid of points with spatial increments corresponding to the Δx and Δy of the regression model:

$$\begin{aligned} x_{i_x} &= x_0 + (i_x - 1)\Delta x; \quad i_x = 1, ..., M_x \\ y_{i_y} &= y_0 + (i_y - 1)\Delta y; \quad i_y = 1, ..., M_y \\ t_{i_t} &= t_0 + (i_t - 1)\Delta t; \quad i_t = 1, ..., M_t \end{aligned}$$

$$\begin{aligned}\zeta_{(i_x,i_y,i_t)} &= \zeta(x_{i_x}, y_{i_y}, t_{i_t}) \\ &= \sum_{j_x=0}^{N_x}\sum_{j_y=0}^{N_y}\sum_{j_t=0}^{N_t}\Phi_{(j_x,j_y,j_t)} \\ &\quad\times \zeta_{(i_x-j_x,i_y-j_y,i_t-j_t)} + \sigma_\epsilon^2\epsilon_{(i_x,i_y,i_t)}\end{aligned} \tag{6}$$

where M_x and M_y define the size of the wave elevation evaluation grid, which is dictated by the size of the domain over which elevations are required and will generally be larger, sometimes far larger, than the length of regression.

The elevation calculation is advanced in time along with the simulation itself. In the application of the autoregressive wave model, the time step of the simulation is matched to the time step of the wave autoregression function. In principle, however, different time steps could be accommodated by either interpolating the wave elevation data in time or performing multiple wave time steps for each simulation time step.

Since the elevation at each point is dependent only on the elevations at lesser or equal x, y, or t, the method is explicit and easily calculated by sweeping through the elevation grid in x and y at each time step. Calculating the elevation on a finite grid presents no major problem—the summation is simply truncated at the edge of the grid.

The required extent of the wave elevation grid will generally be the region over which incident wave data is required plus some allowance at the minimum x and y edges for a "ramp-up" region. For a 3-D potential flow calculation, this is simply the extent of the hull's wetted surface. The issue is a bit more complicated for simulations with forward speed or a significant amount of drift. The 3-D autoregressive wave model is generally cast in a global coordinate system, so the x- and y-grid lines of the evaluation must be inherently fixed in space. Constructing a grid covering the entire range of the simulation would be impractical for a simulation of any length, so a local grid scheme is implemented.

In the local grid scheme, the grid is moved with the ship but grid lines are maintained at integer multiples of the increment grid. In effect, grid lines are added in front of the ship and removed from behind it as the simulation progresses. The addition of grid lines forward of the ship must account for the "ramp-up" time of these added lines. Therefore, the resulting grid must be elongated in the direction of travel. For a typical seakeeping problem with a more-or-less constant speed and heading, the x extent of the grid will be:

$$x_0 = \left(\left\lfloor \frac{(x_g(t) - L/2)}{\Delta x} \right\rfloor - N_x\right)\Delta x \tag{7}$$

$$M_x = N_x + \left(\frac{L + 2UN_t\Delta t}{\Delta x}\right) \tag{8}$$

where $x_g(t)$ is the global x-coordinate of the ship's center (mid-ships) at a given time, L is the ship length, N_x and Δx are the regression order and increment in x, N_t and Δt are the regression order and increment in time, and U is the ship speed; $\lfloor\cdot\rfloor$ is the integer floor function, used to round the grid extents to integer multiples of the grid spacing, so grid lines will be coincident from time step to time step.

For cases with large unsteady motion, including maneuvering in waves and broaching, the grid expansion must consider unsteady speed in both x and y. Figure 1 shows a notional wave evaluation grid (not every grid line is shown) at three simulation time steps for a ship in a slow-speed turn.

4.1 Random White Noise

The term $\sigma_\epsilon^2\epsilon_{(ix,iy,it)}$ in Equation (5) represents a field of white noise. σ_ϵ^2 is the variance of the white noise model and is a scalar value calculated from the regression coefficients described above. Along with the regression coefficients, this value will be constant for stationary waves and a function of time for non-stationary (e.g. rising or falling) seas. The quantity $\epsilon_{(ix,iy,it)}$ is a random function that should have unit variance and the same distribution as the wave elevations. For a Gaussian (normal) distribution, it can be readily approx-

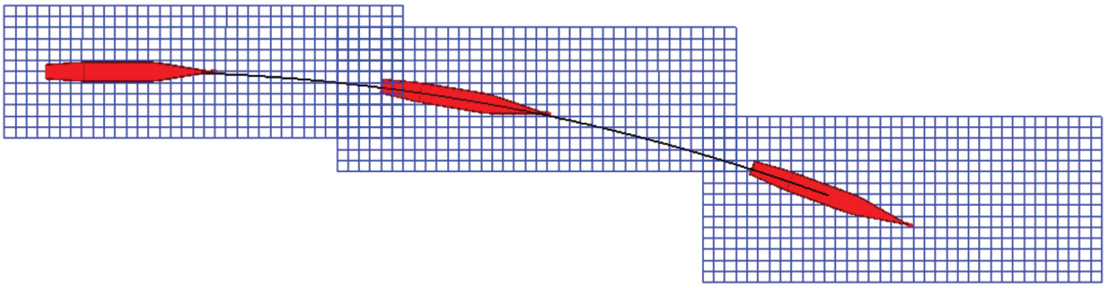

Fig. 1 Moving Elevation Grid for a Low Speed Turn

imated by the expression:

$$\epsilon = \sum_{i=1}^{12} R_i - 6 \tag{9}$$

where R_i is a random value of uniform distribution, and range [0,1], which is the typical value of the intrinsic pseudo-random number function available in most math libraries.

4.2 Repeatability of the Wave Model

In the same way that the "random" phases of the wave components provide different realizations of the irregular wave field in a Longuet-Higgins model, the "randomness" of $\epsilon_{(ix,iy,it)}$ provides independent realizations of the ARM wave field. It is therefore necessary to be able to generate independent sets of these random values.

However, it is also highly desirable to be able to reproduce the identical calculation of the wave field. This is useful for visualizing the motion in waves, post-processing calculations such as relative motion and slamming, or simply repeating a simulation for a specific set of waves. To do this, it is necessary to use a pseudo-random number generator with a seed specification option and to record the size and origin of the regression grid.

4.3 Derivatives of the Elevation Field

Derivatives of the wave elevation in space and time are needed for calculation of the velocity potential field. In an initial implementation, these derivatives are computed using finite difference of the values on the wave elevation grid. In order to allow a central difference calculation of the time derivative, the elevation calculation is run one time step ahead of the simulation. As the implementation of autoregressive continues, the calculation of these derivatives must be evaluated along with the effect and requirements of grid resolution and time step.

5 CALCULATION OF THE INCIDENT WAVE POTENTIAL FIELD

A significant challenge of using the ARM of wave for numerical simulations is that the ARM provides only the elevation field while numerical ship-motion codes generally require the pressure and velocity field beneath these waves. In panel methods, the pressure field is required in order to evaluate the Froude-Krylov forces and the velocity field is required to set up the body boundary condition for the disturbance potential boundary-value problem. In order to address this challenge, the implementation must incorporate an "inverse problem" solver which computes the incident wave velocity potential ($\phi_0(x, y, t)$) beneath the specified wavy surface. This inverse problem solution, which is described in more detail in Degtyarev & Gankevich (2012) and Gankevich & Degtyarev (2015), is summarized below.

The inviscid, incompressible potential flow beneath a free surface is described by the system of equations:

$$\nabla^2\phi = 0,$$
$$\phi_t + \frac{1}{2}|\vec{\nabla}\phi|^2 + g\zeta = -\frac{p}{\rho} \quad \text{on } z = \zeta(x,y,t),$$
$$\frac{D\zeta}{Dt} = \vec{\nabla}\phi \cdot \vec{n} \quad \text{on } z = \zeta(x,y,t), \tag{10}$$

where ϕ is the incident wave potential, D/Dt is the substantial derivative and $\vec{n}$ is the local normal vector to the free surface. The first of these equations satisfies continuity throughout the fluid domain while the second and third are the dynamic and kinematic free-surface boundary conditions, respectively. In the inverse problem, the free surface is known.

5.1 2-D Solution

For unsteady, two-dimensional (x, z, t) flow, (10) can be rewritten as:

$$\begin{aligned} &\phi_{xx} + \phi_{yy} = 0 \\ &\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\zeta = -\frac{p}{\rho} \quad \text{on } z = \zeta(x,t) \\ &\zeta_t + \zeta_x\phi_x = \frac{\zeta_x}{\sqrt{1+\zeta_x^2}}\phi_x + \phi_z \quad \text{on } z = \zeta(x,t). \end{aligned} \tag{11}$$

The 2-D potential at any time can be written as a Fourier transform of a function multiplied by an exponential:

$$\phi(x,z) = \int_{-\infty}^{\infty} E(\lambda) e^{\lambda(z+ix)} d\lambda. \tag{12}$$

This potential implicitly satisfies the continuity equation and can be substituted into the kinematic boundary condition to give:

$$\frac{\zeta_t}{1 - i\zeta_x - i\zeta_x/\sqrt{1+\zeta_x^2}} = \int_{-\infty}^{\infty} \lambda E(\lambda) e^{\lambda(\zeta+ix)} d\lambda. \tag{13}$$

This expression represents a forward bilateral Laplace transform and can be inverted to yield a formula for the coefficients $E(\lambda)$:

$$E(\lambda) = \frac{1}{2\pi i}\frac{1}{\lambda}\int_{-\infty}^{\infty} \frac{\zeta_t}{1 - i\zeta_x - i\zeta_x/\sqrt{1+\zeta_x^2}} \times e^{-\lambda(\zeta+ix)} dx. \tag{14}$$

Substituting (14) into (12) yields the final result:

$$\phi(x,z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\lambda} \left(\int_{-\infty}^{\infty} \frac{\zeta_t}{1 - i\zeta_{x'} - i\zeta_{x'}/\sqrt{1+\zeta_{x'}^2}} e^{-\lambda(\zeta+ix')} dx' \right) \times e^{\lambda(z+ix)} d\lambda. \tag{15}$$

It should be noted that while the free surface must be single valued, the slope of the wave is not assumed to be small, as has been in previous solutions of the inverse problem. Gankevich & Degtyarev (2015) provide a comparison of the previous and present methods.

In the numerical implementation of this scheme for the elevations generated via the autoregressive model, the infinite inner and outer integral limits of (15) are replaced by the corresponding wave surface size $(x_0,\, x_1)$ and wave number interval $(\lambda_0,\, \lambda_1)$ so that the inner integral converges.

The solution of the 3-D problem (2-D spatially + 1-D time) is simular though it, not surprisingly, involves double integrals.

5.2 Estimate and Interpolation of Potential Derivatives

The inverse velocity potential calculation provides the potential on a line of x-points or a grid of (x, y)-points corresponding to the elevation data evaluated from the ARM. Currently, there is no analogous formulae for the fluid velocities, the derivatives of the velocity potential. So derivatives must be calculated using finite difference techniques.

The lateral (x, y) resolution of the velocity potential will be dependent upon the resolution of the wave elevation field. However, in the vertical, (z), direction, the potential can be evaluated for any z, so the resolution and range of the vertical distribution of the potential and its derivatives can be selected based on the requirements of the problem.

6 SUMMARY

Degtyarev & Reed (2011, 2012) presented the development of an autoregression model for incident random waves that is far more computationally efficient than the Fourier series like model of Longuet-Higgins. This model is amenable to modeling the synoptic and temporal processes associated to the development and evolution of ocean waves in a storm.

Degtyarev and Reed also showed that the waves produced by the autoregression model have the correct statistical characteristics spatially and temporally to represent ocean waves—the desired wave spectra can be reproduced and the distributions of physical characteristics is correct. Although the model does not explicitly contain the physics of gravity waves, by using 2- and 3-dimensional (1- or 2-dimensions in space + time) autoregression functions based on actual wave measurements, the model even captures the dispersion relation for gravity waves.

Degtyarev & Gankevich (2012) and Gankevich & Degtyarev (2015) have provided a technique for efficiently computing the velocity potential beneath a prescribed 1-D or 2-D surface, varying with time.

This paper attempts to continue that development by outlining an implementation of an auto-regressive incident wave model for use in a time-domain numerical ship-motion simulation code. Several key aspects of this implementation are described, including the efficient evaluation of the ARM on a set of moving grids for a simulation with steady or unsteady forward speed and the calculation of the incident wave velocity potential field beneath a prescribed wave surface. The latter procedure is not only a critical element of the application of the ARM, but provides a mechanism for implementing other non-traditional ocean wave models in numerical simulations. The complete details of the implementation and examples will be provided in Weems, et al (2016), to be presented later this year.

It remains to be determined whether or not the ARM with the subsequent solution of an initial value problem for the velocity potential beneath the wave surface—the inverse problem, is computationally competitive with a Longuet-Higgins Fourier series based model. However, there certainly will be a point where it is competitive, as the Longuet-Higgins model's speed is inversely related to the number of coefficients required.

Several areas where future research is needed have been identified. One of the most critical appears to be the derivation of a direct method for computing the velocities in the fluid domain, a method similar to that used to compute the velocity potential.

ACKNOWLEDGMENTS

Professor Degtyarev's work with NSWCCD has been supported by the Office of Naval Research Global's Visiting Scientist Program. The authors are grateful to Dr. Woei-Min Lin of ONRG and Dr. Vadim Belenky of NSWCCD for their effort in arranging this effort.

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