Approximation of the non-linear roll damping

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ABSTRACT

The paper discusses how to get the proper estimation of the non-linear damping moment in ship roll with the help of free roll tests. It demonstrates two major points, 1° that the damping moment in terms of approximation is an odd non-analytic function, 2° the standard method based on the ratio of two consecutive amplitudes is of limited meaning for non-linear roll. A new method is proposed, based on approximation of free roll, using the instantaneous values of the logarithmic decrement of damping. It is assumed that the instantaneous values are identical with the equivalent values, obtained from equating work done over one cycle.

KEYWORDS

ship roll, non-linear damping, simulations of ship motions

INTRODUCTION

It is well known that roll damping is non-linear. Damping is needed for simulations of ship motions. In equations of motion a normalized damping is normally used, understood as the ratio between the damping moment and the virtual moment of inertia around the longitudinal axis of rotation. This characteristic value is typically approximated by an odd quadratic polynomial of the form:

$$b_1\dot\phi + b_2\dot\phi|\dot\phi|,$$

the second derivative of which with respect to the speed of roll does not exist at $\dot\phi = 0$. Consequently, this type of approximation is non-analytic.

Damping is an odd function of the speed of roll $\dot\phi$. In general, it seems natural to use odd polynomials for approximating odd functions. In the case of normalized damping, this is an odd polynomial of the speed of roll $\dot\phi$:

$$b_1\dot\phi + b_3\dot\phi^3 + b_5\dot\phi^5 + \ldots,$$

where the coefficients b_1, b_3, b_5, … are constant.

The same applies to other odd functions, as, for instance, the *GZ*-curve, which is an odd function of the angle of heel (roll). Consequently, it should be approximated resorting to odd polynomials (or sine sums) of the angle ϕ, as discussed by Pawłowski (1987).

The above matter seems obvious to mathematicians. Therefore, McCue (2007) in her noteworthy paper did not hesitate at all to use odd polynomials for approximating odd functions. She did this despite the fact that in the original paper she used for reference, non-analytical approximations were employed both for damping and the *GZ*-curve.

BASIC ASSUMPTION

The damping coefficient is normally denoted by *N*. In the case of non-linear damping, this coefficient is amplitude dependent, normally established with the help of free roll tests, varying from oscillation to oscillation. Work dissipated by the damping moment over one cycle during a forced motion can be calculated as follows:

$$L = \int^{2\pi} M\,d\phi = \int^{2\pi} (N_1\dot\phi + N_3\dot\phi^3 + N_5\dot\phi^5 + \ldots)\,d\phi, \tag{1}$$

where the expression in the parentheses is the damping moment *M*. Assuming that the forced motion is harmonic, that is $\phi = a\sin\omega t$, where a is the amplitude of roll, and ω is the circular frequency of oscillation, then $\dot\phi = \omega a\cos\omega t$. Since $d\phi = \dot\phi\,dt$, the following results from equation (1)

$$L = N_1(\omega a)^2 \int^T \cos^2\omega t\,dt + N_3(\omega a)^4 \int^T \cos^4\omega t\,dt + N_5(\omega a)^6 \int^T \cos^6\omega t\,dt + \dots \qquad (2)$$

where $T = 2\pi/\omega$ is the period of oscillations. Introducing notation

$$I_n = \int^T \cos^n\omega t\,dt,$$

the above integrals can be easily calculated by the recurrence equation $I_{n+2} = {}^{n+1}/_{n+2} I_n$, which results from integration by parts. Since $I_0 = T$, the first integral in equation (2) equals $\tfrac{1}{2}T$, the second equals $(\tfrac{1}{2}\cdot\tfrac{3}{4} = \tfrac{3}{8})T$, the third equals $(\tfrac{3}{8}\cdot\tfrac{5}{6} = \tfrac{5}{16})T$, and so on. Hence, equating work done over one cycle yields

$$N_1(\omega a)^2\,\tfrac{1}{2}T + N_3(\omega a)^4\,\tfrac{3}{8}T + N_5(\omega a)^6\,\tfrac{5}{16}T + \dots = N(\omega a)^2\,\tfrac{1}{2}T, \qquad (3)$$

where N is the equivalent linear coefficient of damping, amplitude dependent. For linear damping $N = const$, which means independence of the amplitude of oscillations, whereas for non-linear damping N is a function of the amplitude of roll a. Equation (3) yields an even polynomial relative to the amplitude of roll for the equivalent coefficient of damping:

$$N = N_1 + \tfrac{3}{4}N_3(\omega a)^2 + \tfrac{5}{8}N_5(\omega a)^4 + \dots, \qquad (4)$$

where $N_1 = const$ is a linear part of the equivalent coefficient of damping, independent of amplitude, whereas the other part is non-linear, dependent on the amplitude a. Similar considerations can be found in Błocki (1977, 1980). It is noteworthy that using only two terms in equation (4), frequently found in literature, is insufficient for proper approximation of the non-linear damping, shown later.

A graph of $\nu \equiv \tfrac{1}{2}b$ versus the amplitude a is normally obtained experimentally from free roll tests; in physics the quantity ν is termed the logarithmic decrement of damping. It is assumed that the experimental value of ν is identical with the equivalent one. Having found a polynomial approximation of the logarithmic decrement, equation (4) says how to get the coefficients N_1, N_3, N_5, …, needed in computations. Dividing the above equation throughout by the virtual moment of inertia $J_x + m_{44}$, we get

$$b = b_1 + \tfrac{3}{4}b_3(\omega a)^2 + \tfrac{5}{8}b_5(\omega a)^4 + \dots, \qquad (5)$$

where b_1, b_3, b_5, … are constant, independent of a.

FREE ROLL

It is worth recalling that damping is very difficult to obtain from experiments with reasonable accuracy. Normally, free roll tests are used for this purpose. A typical run of such a roll, carried out at CTO in the 'dry' condition for a ro–pax vessel used for surviving tests, is shown in Figure 1. The scale of the model was 1:50, and vessel's particulars were these:

L_{oa} = 169.90 m	C_b = 0.628
L_{pp} = 159.00 m	h_0 = 3.46 m
B = 28.00 m	m = 16 500 ton
T = 5,73 m	z_G = 12.40 m

Measured values, recorded every 0.02 s with resolution of 0.1° are marked by dots, whereas solid lines correspond to approximated values, based on two different methods, discussed below. Some non-harmonic character of roll, clearly visible for small amplitudes, can be attributed to the presence of water that leaked to the hull during earlier tests of the model in damaged conditions.

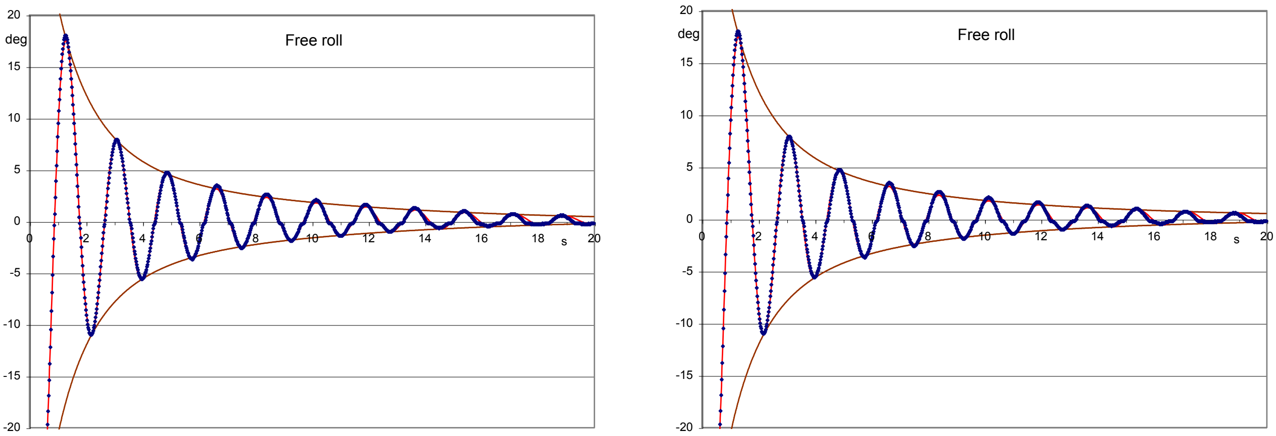

Figure 1. Free roll of a vessel in the intact condition and the run of instantaneous amplitudes

Linear free roll of the ship is described by the equation

$$\phi = \phi_0 + \alpha e^{-\nu t}\cos(\omega t + \varepsilon), \tag{6}$$

where ϕ_0 is a bias (initial heel), α, ν, and ε are constants, $\omega = (\omega_0^2 - \nu^2)^{1/2}$ is the circular frequency of free roll, ω_0^2 equals the coefficient of stability Dh_0 related to the virtual moment of inertia around the longitudinal axis $J_x + m_{44}$, and ν is the logarithmic decrement of damping. The last characteristic value is defined by the equation: $2\nu = N/(J_x + m_{44})$, where N is the damping coefficient. The first two factors in equation (6) can be treated as amplitude at given time instant:

$$a = \alpha e^{-\nu t}. \tag{7}$$

For non-linear free roll with finite initial amplitude, the logarithmic decrement of damping ν varies in the course of time. How it varies, it is not easy to establish, since the problem is ill conditioned. One possibility is to replace the exponent νt in equation (6) and (7) by the integral (cumulative) curve of the logarithmic decrement

$$C = \int_0^t \nu(\tau)\,d\tau = \bar{\nu} t, \tag{8}$$

and approximate C versus time, which can be done through regression. The quantity $\bar{\nu} = C/t$ is the mean cumulative decrement, whereas $\dot{C} = \nu$ is the instantaneous (actual) decrement of damping. As the actual decrement ν and amplitude a are both functions of time, this indirectly defines the instantaneous ν as the function of the instantaneous amplitude of roll a.

Similarly, as the circular frequency of free roll slightly varies in the course of time, the quantity ωt in equation (6) should be also replaced by the integral curve of the circular frequency

$$\Omega = \int_0^t \omega(\tau)\,d\tau = \bar{\omega} t, \tag{9}$$

where $\bar{\omega} \approx (\omega_0^2 - \bar{\nu}^2)^{1/2}$, which can be proved rigorously. In other words, in the case of non-linear free roll ωt is replaced by $\bar{\omega} t$.

APPROXIMATIONS

Various approximations can be used for $C = C(t)$, either by approximating the mean cumulative decrement $\bar{\nu}$, or the actual decrement ν. Best results in both cases give the exponential approximation

$$\nu = \nu_\infty + \beta e^{-\gamma t}, \tag{10}$$

where ν_∞, β and γ are constants, which can be found with the help of the least squares method, using e.g. Solver in Excel. When the mean cumulative decrement $\bar{\nu}$ is approximated, the actual decrement is obtained from the equation $\nu = \dot{C} = {}^d/_{dt}(\bar{\nu} t)$. When the actual decrement ν is approximated, the mean cumulative decrement $\bar{\nu}$ is obtained from equation (8).

The exponential approximation for the mean accumulated decrement is shown in Figure 1 on the left, and for the actual decrement on the right for the same run of free roll. As can be seen, both provide exceptionally good approximations, nearly identical with the real run, proving validity of equation (6) also for non-linear roll, with νt replaced by the integral C.

Graphs of the mean cumulated and actual decrements as the function of time are shown in Figure 2. Curve 1 concerns approximation of the mean decrement, and curve 2 the instantaneous decrement. As can be seen, the two approximations provide almost identical runs of the coefficients $\bar{\nu}$. The same applies to runs of the instantaneous values of ν for about half of time, when amplitudes of roll are large. Afterwards the two curves diverge. Curve 1 for the instantaneous values of ν falls below its asymptotic value, which is wrong. And this can be taken as a rule – approximations of the mean cumulated decrement do not guarantee that the instantaneous decrement will fall monotonically to its asymptotic value. For this reason, it is better to approximate the run of the actual rather than mean decrement. For the latter case the asymptotic value ν_∞ = 0.225/s and for the former ν_∞ = 0.134/s.

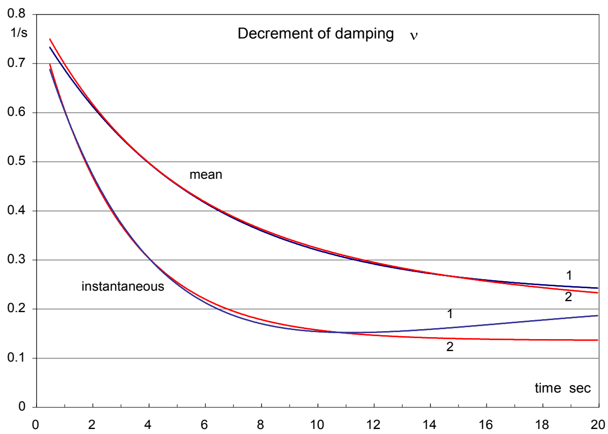

Figure 2. Decrement of roll damping ν as function of time

The resulting prediction of the actual logarithmic decrement ν as the function of the instantaneous amplitude of roll is shown in Figure 3 along with values obtained from the ratio of amplitudes for each cycle, normally used in tests. Using equation (6), with νt replaced by the integral C, yields

$$\nu^* = (1/T)\ln(a_n/a_{n+1}), \tag{11}$$

where T is a period of roll, and a_n/a_{n+1} is the ratio of two consecutive amplitudes, understood as two consecutive extreme values of roll of the same sign. The quantity $\nu^* = (C_{n+1} - C_n)/T$ is nothing other than the mean decrement of damping over one cycle. These values, taken at the average amplitude $½(a_n + a_{n+1})$, are marked with triangles in Figure 3, and approximated by a quadratic curve.

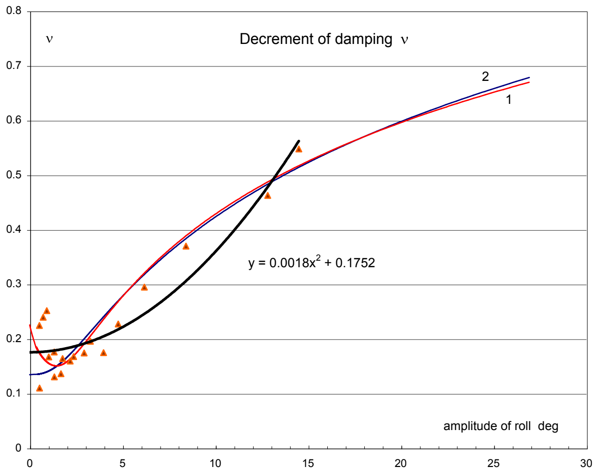

Figure 3. Decrement of damping ν versus roll amplitude along with measured values and a quadratic approximation

As can be seen, the two approximations provide almost identical prediction of the coefficient $\nu = \nu(a)$, except the initial value, well supported by measured values. Curve 1, based on approximation of the mean cumulated decrement has clearly an incorrect run in the neighbourhood of zero, as it falls below the initial (asymptotic) value. On the other hand, curve 2, based on approximation of the instantaneous decrement has an ideal run in the neighbourhood of zero, with vanishing odd derivatives at zero, as in the case of even functions. However, a quadratic approximation is clearly insufficient for that purpose. Things look better, if a biquadratic approximation is used, as shown in Figure 4.

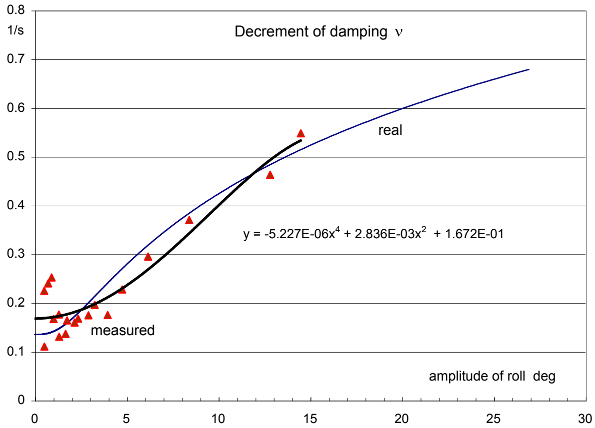

Figure 4. Biquadratic approximation of measured values versus the real run of ν as function of amplitude of roll

The existence of a plateau in the neighbourhood of zero is self-explanatory, if someone realises that for small amplitudes of roll the logarithmic decrement of damping $\nu = const$. On this ground we can expect that all the odd derivatives vanish at zero and the function $\nu(a)$ is even.

Sometimes the coefficient $\nu = \nu(a)$ is found for each half cycle

$$\nu^* = (2/T)\ln(a_{n+½}/a_n), \tag{12}$$

where $a_{n+½}/a_n$ is the ratio of two consecutive amplitudes, understood as two consecutive extreme values of roll, of opposite sign. But this does not help at all. Because measurements of roll are of limited accuracy, a pretty high scatter of points is then obtained, particularly when the amplitude of roll becomes small. Therefore, using half cycles for calculating the coefficient ν is not recommended.

Looking at the measured values someone could think that a linear approximation would be best, as shown in Figure 5, supporting the current generally accepted assumption that damping moment in ship roll is an odd quadratic expression

$$M = N_1\dot{\phi} + N_2\dot{\phi}\,|\dot{\phi}|,$$

where N_1 and N_2 are constants. Ikeda (1978) provides a method for the prediction of the two coefficients.

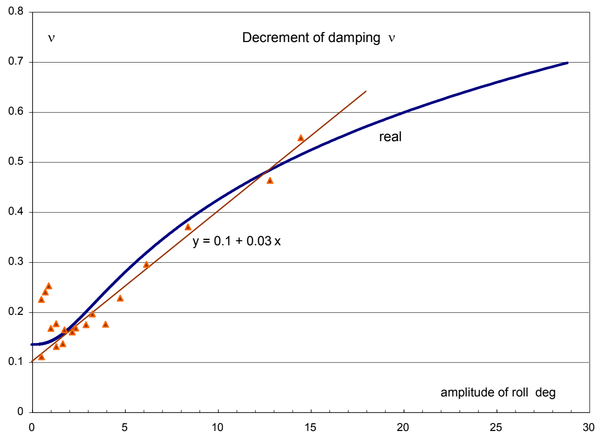

Figure 5. Linear approximation of measured values of ν

Regarding measured values of the actual decrement, using the instantaneous amplitudes we can get the actual values of decrement almost as the continuous function of time. To this end equation (11) should be applied to any two amplitudes taken at time instants far away each other by T seconds. Values calculated this way are shown in Figure 6, which are almost identical with the instantaneous values of decrement ν.

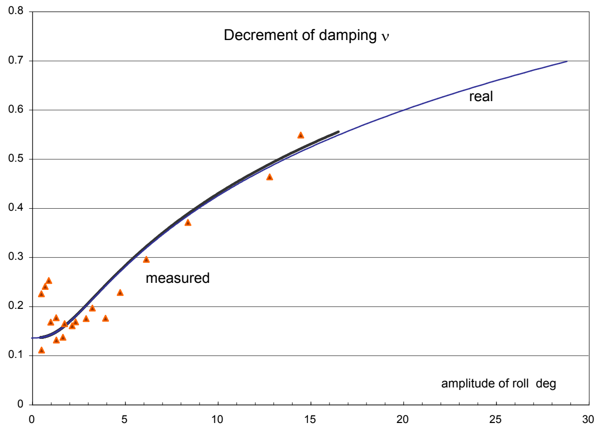

Figure 6. Characteristic and real values of ν

Although the actual decrement is an even function, it is extremely difficult for approximation with the help of even polynomials. Use of three terms, i.e. applying a biquadratic approximation, as shown in Figure 7, is clearly inadequate, whereas using more terms creates numerical problems with definition of high degree polynomials and is undesirable in applications. For that reason we are forced to resort to odd non-analytical polynomials with ||.

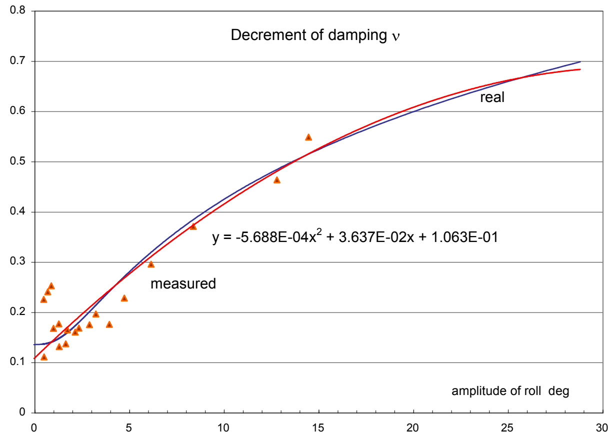

Figure 8. Quadratic approximation of actual values of decrement of damping

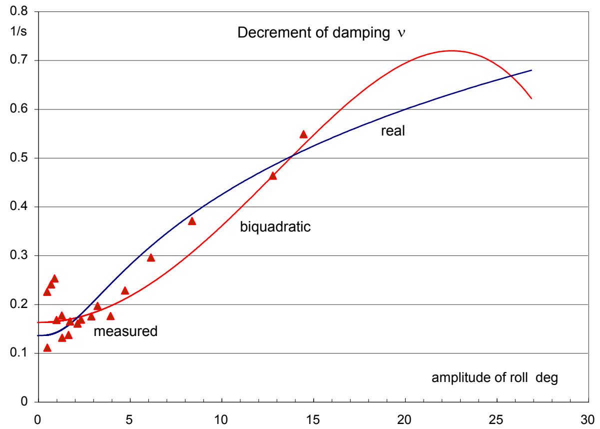

Figure 7. Biquadratic approximation of actual values of decrement of damping

If we abandon the condition of symmetry and use regular polynomials, the instantaneous decrement of damping can be very easily and accurately approximated by quadratic polynomials, as shown in Figure 8. The differences are hardly visible. Numerical quality of approximation is impressive, particularly if compared with Figure 7, though in both cases the same number of terms (three) is used. Application of regular polynomials is contradictory with the basic assumption made earlier, but is necessary due to practical reasons.

REALISTIC ASSUMPTION

Adopting regular polynomials for the coefficient ν = ν(a) is equivalent to the assumption that the damping moment is a non-analytic odd function of the speed of roll $\dot{\phi}$, that can be expanded into a power series, containing even terms with ||

$$M = N_1\dot\phi + N_2\dot\phi|\dot\phi| + N_3\dot\phi^3 + N_4\dot\phi^3|\dot\phi| + N_5\dot\phi^5 + \ldots \tag{13}$$

Equating work done, as before, over one cycle yields

$$N_1(\omega a)^2\, {\textstyle\frac{1}{2}}T + N_2(\omega a)^3\, \tfrac{4}{3\pi}T + N_3(\omega a)^4\, \tfrac{3}{8}T + N_4(\omega a)^5\, \tfrac{16}{15\pi}T + N_5(\omega a)^6\, \tfrac{5}{16}T + \ldots = N(\omega a)^2\, \tfrac{1}{2}T, \tag{14}$$

Recalling the previous recurrence identity, it is easy to find the coefficients at even damping coefficients N_2, N_4, N_6, …. Since $I_1 = \frac{2}{\pi}T$, the coefficient at N_2 equals $\frac{2}{3}\cdot\frac{2}{\pi} = \frac{4}{3\pi}$, at N_4 equals $\frac{4}{5}\cdot\frac{4}{3\pi} = \frac{16}{15\pi}$, at N_6 equals $\frac{6}{7}\cdot\frac{16}{15\pi} = \frac{32}{35\pi}$, and so on.

Equation (14) yields a polynomial for the equivalent coefficient of damping relative to the amplitude of roll:

$$N = N_1 + \tfrac{8}{3\pi}N_2(\omega a) + \tfrac{3}{4}N_3(\omega a)^2 + \tfrac{32}{15\pi}N_4(\omega a)^3 + \tfrac{5}{8}N_5(\omega a)^4 + \ldots, \tag{15}$$

In most cases it is sufficient to take four terms in the above expansion. Dividing the above equation throughout by the virtual inertia $J_x + m_{44}$, we get

$$b = b_1 + \tfrac{8}{3\pi}b_2(\omega a) + \tfrac{3}{4}b_3(\omega a)^2 + \tfrac{32}{15\pi}b_4(\omega a)^3 + \tfrac{5}{8}b_5(\omega a)^4 + \ldots, \tag{16}$$

where b_1, b_2, b_3, … are constant, independent of a. As dimension of b is 1/s, the same dimension has the coefficient b_1, the coefficient b_2 has no dimension, dimension of b_3 is s, b_4 is s^2, b_5 is s^3, and so on. Because $b = 2\nu$, therefore

$$\nu = \tfrac{1}{2}b_1 + \tfrac{4}{3\pi}b_2(\omega a) + \tfrac{3}{8}b_3(\omega a)^2 + \tfrac{16}{15\pi}b_4(\omega a)^3 + \tfrac{5}{16}b_5(\omega a)^4 + \ldots, \tag{17}$$

A graph of ν is needed from free roll tests, as shown in Figure 8. In this case a two-degree approximation fits almost perfectly the run of instantaneous values of the decrement ν.

USE OF THE APPROXIMATION

Knowing polynomial approximation of the experimental decrement ν relative to the amplitude of roll, as shown in Figure 8, the coefficients of damping b_1, b_2, b_3, … can be easily defined by comparing expansion (17) with the approximation of ν. By doing so, we have to remember that the amplitude a in equation (17) is in radians, whereas in Figure 8 – in degrees. Hence,

$$\tfrac{1}{2}b_1 = 0.10632/\mathrm{s}$$
$$\tfrac{4}{3\pi}b_2\omega = 0.036370\cdot(180/\pi)/\mathrm{s},$$
$$\tfrac{3}{8}b_3\omega^2 = -0.00056883\cdot(180/\pi)^2/\mathrm{s},$$

The mean circular frequency for the model investigated equals $\omega = 3.54$/s. Hence, the coefficients b for the model are, as follows

$$b_1 = 0.21264/\mathrm{s},$$
$$b_2 = 1.3870\ \mathrm{s},$$
$$b_3 = -0.39737\ \mathrm{s}^3.$$

For the ship, they have to be rescaled according to the laws of modeling. Since the model is in the scale 1:50, one second in real scale is $50^{1/2} \approx 7.07$ longer than in model scale. Therefore, the coefficients b for the vessel are, as follows

$$b_1 = 0.0301/\mathrm{s},$$
$$b_2 = 1.3870\ \mathrm{s},$$
$$b_3 = -2.8098\ \mathrm{s}^3.$$

The virtual moment of inertia for the ship around the longitudinal axis equals $J_x + m_{44} = 2\,204\,569$ ton m^4. Hence, the two first damping coefficients are these: $N_1 = 66\,295$ ton m^4/s, and $N_2 = 3\,057\,718$ ton m^4.

According to Ikeda, the two values are, as follows: $N_1 = 61\,900$ ton m^4/s, and $N_2 = 1\,120\,682$ ton m^4. They amount to 93.4% and 36.7% of the real values. In model scale, they correspond to $b_1 \approx 0.199$/s, $b_2 \approx 0.508$. Such coefficients give a straight line in Figure 5 described by the equation: $\nu = 0.0993 + 0.0133\,a$. It crosses the ordinate axis at a point $\nu \approx 0.099$/s, practically the same as for the subject model, but with inclination merely 44.3% (more than twice smaller) of the inclination for the linear regression, shown in Figure 5. It happens despite the fact that the Ikeda's coefficient N_2 includes the effect of bilge keels.

Knowing the damping moment, it is easy now to get the equation for free roll

$$(J_x + m_{44})\ddot\phi + M + Dh_0\phi = 0,$$

where M is the damping moment, given by equation (13). Dividing it throughout by the virtual moment of inertia $J_x + m_{44}$, we get

$$\ddot\phi + b_1\dot\phi + b_2\dot\phi|\dot\phi| + b_3\dot\phi^3 + b_4\dot\phi^3|\dot\phi| + b_5\dot\phi^5 + \omega_0^2\phi = 0, \tag{18}$$

where b_1, b_2, b_3, b_4, b_5 are the normalized coefficients of damping, as derived above. The angle ϕ is in radians. To get roll simulations in degrees, the angle ϕ should be replaced by $\phi \cdot \pi/180$.

It is worth noting that the virtual moment of inertia $J_x + m_{44}$ corresponds to a virtual (physical) axis of rotation, located at the virtual ship mass centre (the mass centre for the ship along with the added mass in sway), as discussed by Balcer (2004). For the ship investigated, the virtual axis lies 2.79 m below the ship centre of gravity.

Using free roll tests we get the virtual moment of inertia related to the virtual axis of rotation. By calculations, this quantity value is provided normally for the axis passing through the mass centre of the ship. If this is the case, we have to remember to transform it to the virtual axis.

The above coefficients of damping have been derived based on equation (16), valid for a forced harmonic roll with constant amplitude of roll. Here arises a question, if they are valid for free roll, with gradually decaying amplitude and damping? For free roll equation (16) is still valid provided that we take the mean values for *a* and *b* at given cycle. A graph of the mean decrement of damping ν versus the mean amplitude *a* is, however, exactly the same as graphs based on instantaneous values, termed 'real', shown on the previous figures. On this basis we can expect that damping coefficients are valid not only for free roll but also for roll in natural conditions.

CONCLUSIONS

Based on the results and arguments presented in this paper the following conclusions can be drawn:

- from the theoretical point of view, the damping moment is an odd *analytic* function, which is, however, difficult to expand into a power series, containing odd terms only
- in terms of approximation the damping moment behaves as if it was an odd *non-analytic* function that can be neatly expanded into a power series, containing even terms with $||$
- the standard method for definition of ν, based on the ratio of two consecutive amplitudes is correct also for non-linear roll, but of limited meaning
- approximation of free roll can provide robust values for the instantaneous decrement ν as a function of the instantaneous amplitude of roll
- assuming that the instantaneous decrement is identical with the equivalent value, obtained from equating work done over one cycle, allows for definition of the coefficients of non-linear damping, needed in simulations of ship motions.

ACKNOWLEDGEMENTS

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