# Assessment of Short-Term Risk with Monte-Carlo Method

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## ABSTRACT

The paper describes a method for the direct assessment of the probability of partial stability failure for an intact ship. The method is essentially a statistical extrapolation, allowing explicit account of influence of nonlinearity of GZ curve on roll distribution. It is achieved by using a Peaks-Over-Threshold method for extrapolation. The method is also capable of simultaneously treating large port and starboard roll angles. To avoid possible inapplicability of Poisson flow, an envelope approach is used. A partial stability failure is associated with the upcrossing of the dangerous level by the envelope. The proposed method is called "Envelope Peaks over Threshold" (EPOT). Application of EPOT is demonstrated with simulated wave elevations.

## KEYWORDS

Problem of Rarity, Principle of Separation, Partial Stability Failure, Statistical Extrapolation

## INTRODUCTION

The main principle that allows solving the problem of rarity is separation. Instead of one problem with very rare events, two or more related problems are considered: "non-rare" and "rare". The "non-rare" problem is crossing a threshold that is low enough that a statistically significant number of crossings can be observed in a model test or numerical simulation. The "rare" problem is a statistical extrapolation of the data above this threshold, see Figure 1.

Nonlinearity is accounted for by separating small and large-amplitude motions with the threshold. If any sort of statistical fit is used on roll motion data in its entirety, the resulting fit will be dominated by the small-amplitude motions where the roll motion is still relatively linear, and the influence of nonlinearity will generally be not represented properly. The threshold must therefore be high enough, so that the influence of nonlinearity above that threshold can be considered substantial. It cannot be chosen based purely on statistics. Physical considerations based on the shape of the GZ curve must be included as well, however setting particular limits on a threshold is outside of scope of this paper; these limits are assumed to be given.

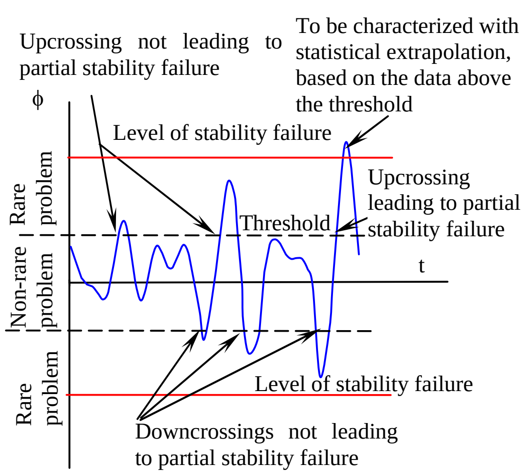


Figure 1 Summary of the current method: separation principle

## BOTH-SIDES CROSSING

Partial stability failure in a form of a large roll event is equally dangerous on either side of a ship. Therefore, a random event of upcrossing is not yet a complete model of partial stability failure. A complete model of the partial stability failure should include both upcrossing of a specified level on the positive side and downcrossing of the specified level on the negative side. This random event can be written as:

$$X = \left((\phi(t) < a) \cap (\phi(t+dt) \geq a)\right) \cup \left((\phi(t) > b) \cap (\phi(t+dt) \leq b)\right) \quad (1)$$

Here $X$ is a random event associated with partial stability failure; $a$ is a positive level of exceedance and $b$ is negative level of exceedance. Obviously, if the mean value of roll is zero and requirements are the same for the both sides:

$$if\left(m(\phi) = 0\right) \Rightarrow a = -b \quad (2)$$

If the distribution of the roll and roll rate are symmetric, the rate of both-sides crossings can be expressed as:

$$\lambda_{ab} = 2f(a)\int_0^{\infty} f(\dot{\phi})\dot{\phi}d\dot{\phi} \quad (3)$$

In particular, for the generic normal process $x(t)$:

$$\lambda_{ab} = \frac{1}{\pi}\sqrt{\frac{V_{\dot{x}}}{V_x}}\exp\left(-\frac{a^2}{2V_x}\right) \quad (4)$$

Where $V_x$ is the variance of the process and $V_{\dot{x}}$ is the variance of it derivative.

The random event of both-side crossing does not follow Poisson flow, as the independence condition is difficult to meet. The autocorrelation function will not die out during the half a period; so if there was an upcrossing through the positive thresholds, the downcrossing through the negative threshold is more likely.

## ENVELOPE APPROACH

The ability to apply Poisson flow is important, as it is difficult to provide an explicit relationship with time outside of the Poisson flow assumption. Belenky & Breuer (2007) used the envelope of the roll process to overcome similar difficulty while dealing with parametric roll; such a process usually has a very narrow spectrum. The narrow spectrum results in significant clustering (or grouping) of the high peaks. As a result, even one-sided upcrossings become dependent on neighboring cycles, as once upcrossing occurs, it is very likely that it will occur again on the next period of motion.

The envelope $a(t)$ is defined as

$$a(t) = \sqrt{\phi^2 + \psi^2} \quad (5)$$

Where $\psi$ is a complimentary process that can be obtained with Hilbert transform.

An additional difficulty here is that the spectrum of roll motions is not necessarily narrow and the envelope cannot be considered as a slowly changing function. In some cases this can result in the envelope peaking higher than the process itself (due to the behavior of the complimentary process). The envelope can then cross the level of interest while the process does not, see Figure 2.

To avoid this artificial crossing, the piecewise linear approximation of the envelope is used, also shown in Figure 2. Values for this "peak-based" envelope are calculated using linear interpolation between the absolute values of peaks or zero-crossing peaks of the process. Using absolute values ensures that both-sides crossing are taken into account as opposed to just upcrossing. This approach is also helpful while dealing with relatively narrow-banded processes, such as ship motion in following and stern-quartering seas.

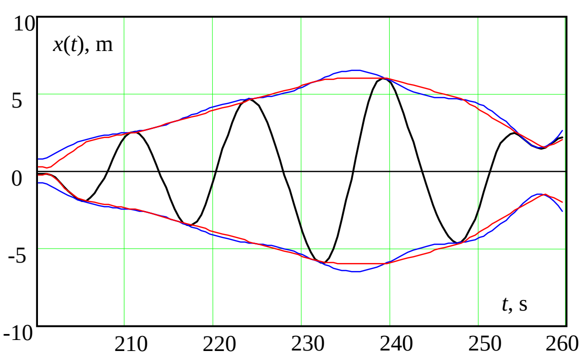


Figure 2. Zoomed in envelope (blue) peak-based or piece-wise linear approximation of the envelope (red) evaluated for wave elevations (Bretshneider spectrum at typical sea state 8)

**ENVELOPE PEAK OVER THRESHOLD**

The main challenge that the problem of rarity poses comes from the nonlinear nature of large-amplitude roll motions. It is well known that large-amplitude roll motions cannot, in general, be characterized by normal distribution (Belenky & Sevastianov, 2007). The type of distribution depends strongly on the shape of the ship's righting arm curve, which may change significantly in waves. It is also difficult to fit a distribution with simulated or measured data; because only the large-amplitude motions carry information on the nonlinearity of the motion and they are rare.

For the same reason it is also difficult to fit the extreme value distribution. Because the dynamical system possesses significant nonlinearity, any statistical fit based on all the data may be misleading, as these data may be dominated by relatively mild nonlinearity. The resulting distribution fit may not reflect the physical properties of the dynamical system for large displacements.

The envelope-peaks-over-threshold method enables the implementation of the principle of separation and avoids the inapplicability of Poisson flow. Then, the probability of at least one large roll event during time $T$ is as follows:

$$\begin{aligned} &P(T \mid E > a_2) = 1 - \exp(-\lambda T) \\ &\lambda = \xi \cdot P(E > a_2 \mid E > a_1) \end{aligned} \qquad (6)$$

Here $\xi$ is the rate of upcrossing through the threshold $a_1$, while $a_2$ is the level of stability failure.

The objective of the non-rare problem is finding the rate of upcrossing, $\xi$, of a given threshold, $a_1$, by the peak-based envelope. The objective of the rare problem is to find conditional probability, $P(\phi > a_2 \mid \phi > a_1)$, that the envelope exceeds the level of partial stability failure, $a_2$, once a given threshold, $a_1$, is crossed.

The value of the threshold plays an important role in separating small and large-amplitude motions. The threshold must therefore be high enough, so that the influence of nonlinearity above that threshold can be considered substantial.

**NON-RARE PROBLEM**

The most direct way to estimate upcrossing of the peak-based envelope is direct counting; then the mean number of events can be estimated as:

$$m_U^* = \frac{1}{N_R}\sum_{j=1}^{N_R} N_{Uj} \qquad (7)$$

Where $N_{Uj}$ is the number of events observed during record $j$. The estimate of rate of upcrossing is:

$$\xi^* = \frac{m_U^*}{T_R} \qquad (8)$$

Where $T_R$ is duration of the record.

The confidence interval for estimate (8) can be found using auxiliary random variable (Kramer & Leadbetter 1968):

$$U_{i,j} = \begin{cases} 1 & E_{i,j} \le a_1 \cap E_{i+1,j} > a_1 \\ 0 & Otherwise \end{cases} \qquad (9)$$
$$i = 1,..,n\,; \quad j = 1,...,N_R$$

If the upcrossings are independent, this auxiliary random variable has a binomial distribution with parameter $p$ – probability that an upcrossing occurs in a particular time instant. It can be estimated as:

$$p^* = \frac{1}{nN_R}\sum_{i=1}^{n}\sum_{j=1}^{N_R} U_{i,j} \qquad (10)$$

The number of upcrossings observed during record $j$ can be expressed though this auxiliary variable as:

$$N_{Uj} = \sum_{i=1}^{n} U_{i,j} \qquad (11)$$

The number of upcrossings can be related to the estimate of the upcrossing rate. $N_{Uj}$ is the sum of independent variables with a binomial distribution, each of which has the same parameter, $p$. This sum also has a binomial distribution with the same parameter $p$, but with n equal to the sum of the number of cases (time steps).. In the case of $N_R$ records, the total number of cases becomes:

$$N = N_R \cdot n \tag{12}$$

Then, the probability that $N_R$ records, each with $n$ time steps, will contain $k$ upcrossings can be expressed as:

$$P(k) = \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k} \tag{13}$$

Formula (13) also can be interpreted as the probability mass distribution for the number of upcrossings for all the records. The number of upcrossings $k$ is related to the estimated rate of upcrossing as:

$$\xi^* = \frac{k}{T_R n} \tag{14}$$

Boundaries of confidence interval for $\xi^*$ are expressed as:

$$\xi^*_{low} = \frac{1}{T_R n} Q\left(\frac{1-\beta}{2}\right); \quad \xi^*_{up} = \frac{1}{T_R n} Q\left(\frac{1+\beta}{2}\right) \tag{15}$$

Where $Q(P)$ is an inverse to the cumulative distribution function (CDF) for $P(k)$ and $\beta$ is a given confidence probability. $Q(P)$ is often referred to as the Quantile function.

## RARE PROBLEM: DIRECT FIT

The objective of the rare problem is to find the probability of the envelope crossing the given level of stability failure, $a_2$, if the threshold, $a_1$, was already exceeded. This probability can be trivially found if the distribution of envelope peaks over the threshold is known:

$$P(E > a_2 \mid E > a_1) = 1 - F(E_m \mid E_m > a_1) \tag{16}$$

Here $F(E_m| E_m > a_1)$ is the CDF of the envelope peaks over the threshold (see Figure 3) . It can be found through a Weibull fit to the available statistical data, using the method of moments or the maximum likelihood method (Cohen 1965), see Figure 4.

Both figures use a dataset of wave elevations simulated with a Bretshneider spectrum for a typical sea state 8.

$$f(x) = \begin{cases} \frac{k}{\alpha}\left(\frac{x-\theta}{\alpha}\right)^{k-1} \exp\left(-\left(\frac{x-\theta}{\alpha}\right)^k\right) & x \ge \theta \\ 0 & x < \theta \end{cases} \tag{17}$$

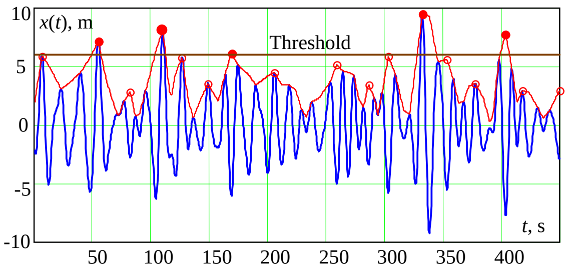


Figure 3 Envelope Peaks over Threshold (filled circles)

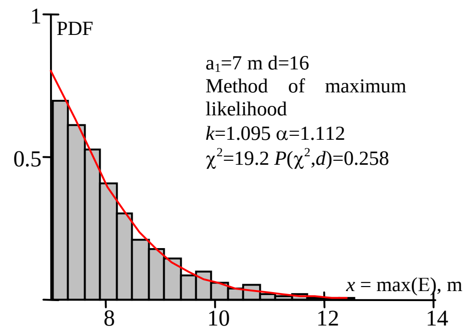


Figure 4 Weibull Fit for Envelope Peaks over Threshold

The width of bins for the histogram in Figure 4 was calculated with the following formula (Scott, 1979):

$$W = \frac{3.5\sigma}{\sqrt[3]{N_p}} \tag{18}$$

Where $\sigma$ is standard deviation and $N_p$ is the number of available data points.

Both distribution fitting methods use statistical data to find the parameters of the distribution (17). Therefore these parameters are random values, which mean the rate of upcrossing is also a random number. The confidence interval must therefore be evaluated to reflect statistical uncertainty. In fact, the easiest way to evaluate the confidence interval for the upcrossing rate is to compute it for the distribution (17) using the method described in (Belenky & Weems, 2008); sample results are shown in Figure 5 and Figure 6. The confidence interval widens as the threshold is raised since there are less data points available.

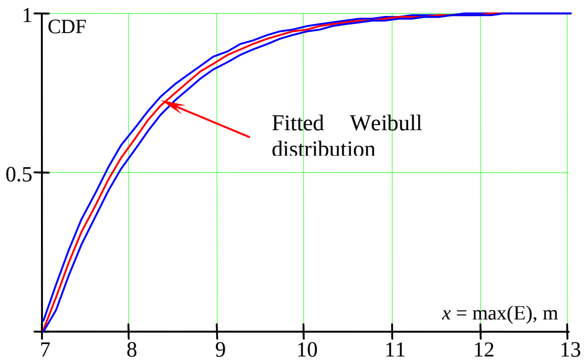


Figure 5 Weibull CDF with Confidence Interval Fitted for Envelope Peaks Exceeding the Threshold of 7 m

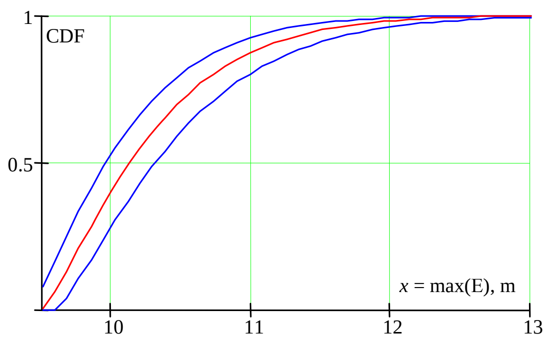


Figure 6 Weibull CDF with Confidence Interval Fitted for Envelope Peaks Exceeding the Threshold of 9.5 m

As a result, it is possible to propagate statistical uncertainty throughout the method and obtain the final result (6) with a confidence interval.

A series of results for these calculations done for 200 simulated records of wave elevations of 30 min durations each (Bretshneider spectrum at typical sea state 8, with significant height 11.5 m and modal period 16.4 sec) was calculated for different threshold values and are shown in Figure 7.

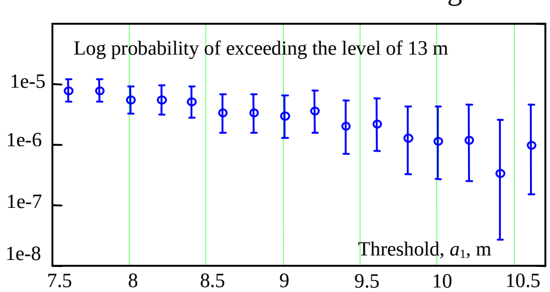


Figure 7 Statistical Extrapolation of Upcrossing Rate

Figure 7 shows some variability of the extrapolated estimate. As the threshold increases, the estimate shows some decrease, while the confidence interval becomes wider.

In principle there can be two tendencies affecting the result. The accuracy of the Weibull fit is better for extrapolation if the data points are closer to the target, but the uncertainty is larger as there are fewer and fewer data points available. The optimum is achieved somewhere in the middle. Therefore averaging the results from different thresholds may be useful:

$$\lambda_a = \frac{1}{N_{a1}} \sum_{i=1}^{N_{a1}} \lambda(a_{1i}) \tag{19}$$

As the first expansion, averaging was also applied to the boundaries of the confidence intervals:

$$\begin{aligned} \lambda_a^{low} &= \frac{1}{N_{a1}} \sum_{i=1}^{N_{a1}} \lambda^{low}(a_{1i}) \\ \lambda_a^{up} &= \frac{1}{N_{a1}} \sum_{i=1}^{N_{a1}} \lambda^{up}(a_{1i}) \end{aligned} \tag{20}$$

## RARE PROBLEM: EXTREME VALUE FIT

The solution of the rare problem involves the evaluation of the probability using the tail of the distribution. Difficulties with predicting the behavior of the tail of fitted distributions are not new. These difficulties were one of the motivations for the development of extreme value theory; therefore it is quite logical to try to use extreme distributions for the rare problem. In its classic interpretation, the extreme value distribution describes probabilistic properties of an extreme value observed during a given time.

To fit an extreme value distribution a time window $T_W$ is introduced; the largest value observed during this time represents one data point, see Figure 8.

The Weibull distribution can be fit using these data points. The resulting distribution will be a conditional distribution, as only points above the given thresholds are used. By the definition of the cumulative distribution function:

$$F_{EV}(a_2 \mid a_1, T_W) = P(E \le a_2 \mid E > a_1, T_W) \qquad (21)$$

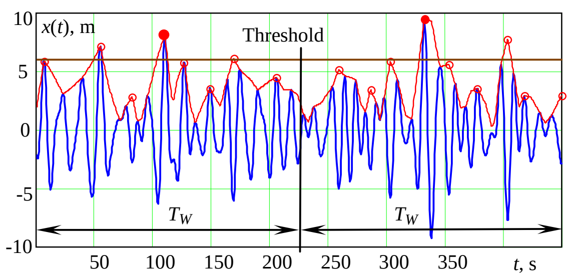


Figure 8 Data Points for Extreme Value Distribution of Envelope

The probability of exceedance of the level $a_2$ during time $T$ is expressed as:

$$P(E > a_2 \mid T_W) = P(E > a_1 \mid T_W) \cdot P(E > a_2 \mid E > a_1, T_W) \qquad (22)$$

Here $P(E>a_1|T_W)$ is the probability of at least one exceedance of the given threshold, while $P(E>a_2|\, E>a_1, T_W)$ is the conditional probability of an exceedance of the level $a_2$ once the threshold $a_1$ has been crossed. The latter is a probability of a random event, complimentary to (21) and therefore it can be expressed through conditional CDF as:

$$P(E > a_2 \mid E > a_1, T_W) = 1 - F_{EV}(a_2 \mid a_1, T_W) \qquad (23)$$

The probability of at least one exceedance of the given threshold can be expressed using Poisson flow, as the rate of upcrossing through the threshold $a_1$ is the solution of the non-rare problem:

$$P(E > a_1 \mid T) = 1 - \exp(-\xi T_W) \qquad (24)$$

A similar expression can be written for the probability of at least one exceedance (or upcrossing) of the level $a_2$:

$$P(E > a_2 \mid T) = 1 - \exp(-\lambda T_W) \qquad (25)$$

The rate of events $\lambda$ is the final objective; substitution of equation (23-25) into (22) allows expressing it through the extreme value CDF:

$$\lambda = -\frac{1}{T_W}\ln\Big(\exp(-\xi T_W) + \big(1-\exp(-\xi T_W)\big)F_{EV}(a_2 \mid a_1, T_W)\Big) \qquad (26)$$

Taking into account (6) the solution for the rare problem (independent of time of exposure, $T$) is expressed as:

$$P(E > a_2 \mid E > a_1) = -\frac{1}{\xi T_W}\ln\Big(\exp(-\xi T_W) + \big(1-\exp(-\xi T_W)\big)F_{EV}(a_2 \mid a_1, T_W)\Big) \qquad (27)$$

The result of sample calculations are shown in Figure 9 and, as expected, the variability of is less (at least visually) in comparison with the fit of Weibull distribution to peaks, as shown in Figure 7. Nevertheless using an averaging procedure (19-20) seems to be reasonable for this case as well.

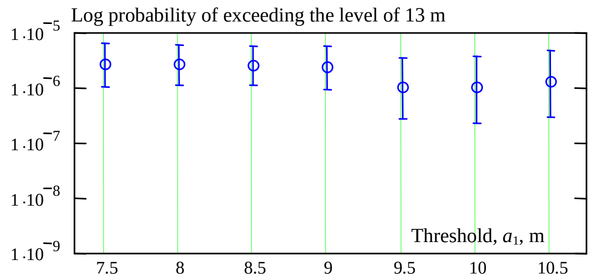


Figure 9 Statistical Extrapolation of Upcrossing Rate Using Extreme Value Distribution Fit for Rare Problem

## COMPARISON WITH THEORETICAL SOLUTION

Simulated wave elevations were used as a numerical example. As this is a normally distributed process the theoretical solution may exist.

However, the failure event is associated with an upcrossing of the peak-based envelope through a certain level. The probability of this event cannot be exactly expressed in closed form, as there is a subtle difference between the peak-based envelope and theoretical envelope defined by formula (5).

Nevertheless, for a relatively high level of upcrossing, the difference between the probability of upcrossing of the theoretical envelope and the peak-based envelope may not be that significant, as a large peak of the process belongs to both the theoretical and peak-based envelopes. Therefore the first candidate for the theoretical solution is the rate of upcrossing of the theoretical envelope

$$\lambda_e = a\sqrt{\frac{(\omega_2^2 - \omega_1^2)}{2\pi V_x}}\exp\left(-\frac{a^2}{2V_x}\right) \qquad (28)$$

Where $\omega_1$ is the mean frequency, $\omega_2^2$ is the second moment of the spectral area, normalized by the variance of the process $V_x$; $a$ is the level of crossing. The derivation of this formula is trivial as the distribution of the envelope is Rayleigh and its derivative is normal.

For the very same reason, the Rayleigh distribution can be assumed for the rare solution. The upcrossing rate in the non-rare solution can be approximated as:

$$\xi = \exp\left(c_0 + c_1 a_1 + c_2 a_1^2\right) \qquad (29)$$

For the purpose of numerical example the coefficients $c_0$ $c_1$ $c_2$ are evaluated from statistics with a least-squares method.

For the very large level of crossings, it may also be possible to use formula (4); it may be so rare that crossing occurs only on one side.

All three these theoretical solutions, nevertheless, remain approximations. However comparisons with extrapolation results may be used as a very coarse verification. The comparison is shown in Figure 10 and confirms the ability of the proposed method to yield reasonable predictions with statistical extrapolation.

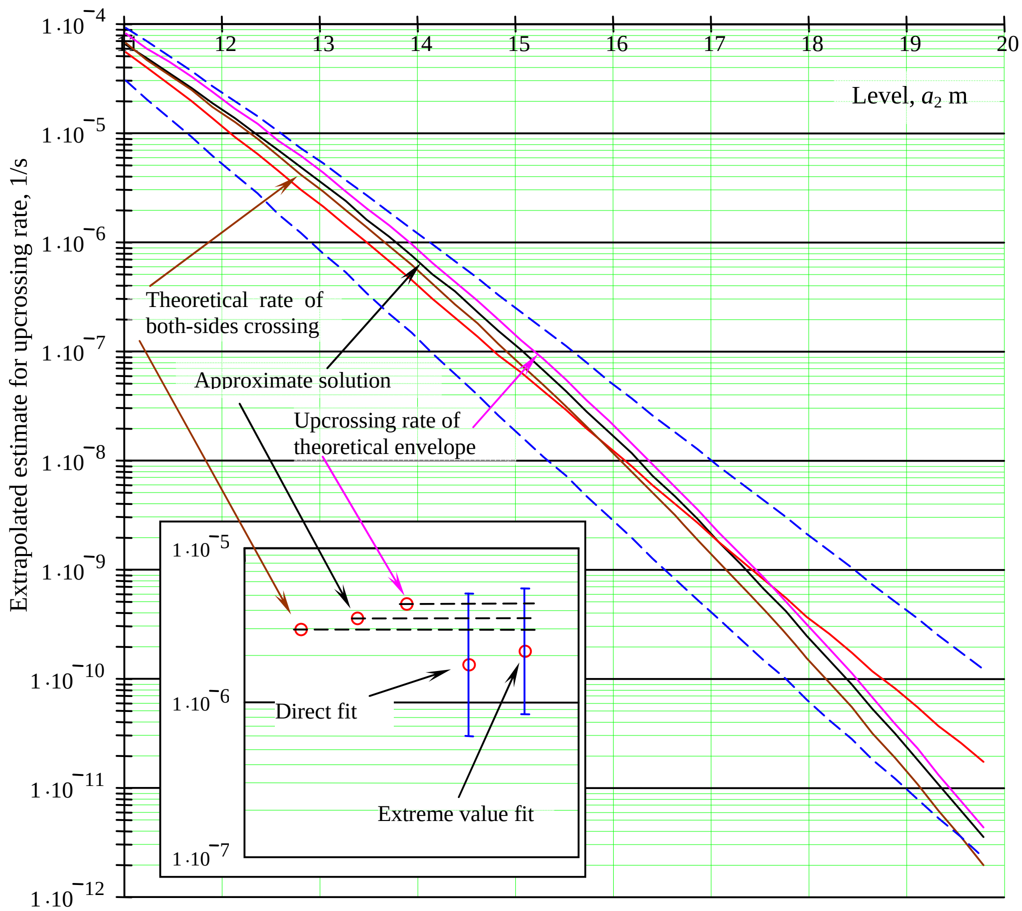


Figure 10 Averaged estimate of rate of upcrossing of the peak-based envelope extrapolated using extreme value distribution. Insert shows the extrapolation for the level of 13 m using both direct and extreme value fit of Weibull distribution.

**SUMMARY**

The EPOT method offers several advances in the context of the direct assessment of partial stability failure. The use of the data exceeding the threshold accounts for the non-linearity of a ship's roll motion as the linear portion of the response does not dominate the distribution fit. Use of the envelope handles both port and starboard rolls ensuring the applicability of the Poisson Flow. The envelope also accounts for the dependence of subsequent roll cycles; this is important for narrow banded processes, such as roll motion of a ship operating in following and quartering seas.

The EPOT method can be used for the direct assessment of stability failures for ships. It may be used with simulation data as well as model tests data.

The EPOT method is still under development. Future work includes evaluating the performance of the algorithm when non-linearity of the roll response becomes severe, such as happens near the peak of the righting arm curve.

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